

A Branch-and-Price-and-Cut Method For Ship Scheduling with Limited Risk

Hee-Su Hwang
Jay M. Rosenberger
Department of Industrial and Manufacturing Systems Engineering
P.O. Box 19017
University of Texas at Arlington
Arlington, TX 76019-0017

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Abstract

Traditional ship-scheduling models ignore uncertainty, even in highly volatile markets. We present a set-packing model that limits risk using a quadratic variance constraint. After generating first-order linear constraints to represent the variance constraint, we develop a branch-and-price-and-cut algorithm for medium-sized ship-scheduling problems. Computational results are provided, and extensions are discussed.

Risk is prevalent in today's global economy. Threats of terrorism, depleted oil reserves, and war are examples of new sources of instability. Incomes in the United States have become more volatile, bankruptcies are more frequent, and households are shouldering more uncertainty (Gosselin, 2004). As economic fluctuation increases, logistics planners need to find methods of managing risk. We modify here a traditional set-packing problem for ship scheduling by adding a quadratic constraint to limit the variance of profit. The new model is formulated as

$$\begin{aligned} & \max cx \\ & \text{s.t. } Ax \leq 1 \\ & \quad x^T Qx \leq d \\ & \quad x \in \{0, 1\}^n, \end{aligned}$$

where A is a 0-1 matrix, and Q is a symmetric positive definite matrix in which an entry in the matrix $[q_{ij}]$ is the covariance of profit for selecting both sets i and j . Although the application considered in this paper is commercial ship-scheduling, the use of the set-packing problem is prevalent in industry. Some applications include air traffic flow management (e.g., Rossi and Smriglio, 2001), aircraft rescheduling (e.g., Andersson and Vabrand, 2004), and plant location (e.g., Cho et al., 1983; Canovas et al., 2002). Moreover, a set-packing problem can be easily transformed into a set-partitioning problem, and some set-partitioning applications include commercial airline crew scheduling (e.g., Vance et al., 1997; Klabjan et al., 2001), aircraft rerouting (e.g., Rosenberger et al., 2003), vehicle routing (e.g., Sindhuchao, 2005; Desaulniers, 2003), political redistricting (Mehrotra, 1998), and organ transplantation (Kong, 2005).

In this paper we formulate a new ship-scheduling model as a set-packing problem with a quadratic constraint that limits the variance of shipping profit. Then we develop a solution method that generates columns and cuts in a branch-and-bound tree. A brief background on the need for managing profit fluctuations in the shipping business is given in the next section. In Section 2, the ship-scheduling model that limits variance is described. Solution approaches based on a branch-and-price-and-cut method are then developed in Section 3, and computational results are presented in Section 4. Finally, conclusions and future research are discussed.

1 Background and Literature

Ship-scheduling models optimize the transportation of commodities, so they are vital to world trade and military logistics. A ship requires a multi-million dollar capital investment, and the daily operating costs of a ship can be tens of thousands of dollars. Consequently, improved fleet utilization can yield significant financial benefit.

Since the pioneering work of Dantzig and Fulkerson (1954), ship-scheduling models have been studied extensively in academic literature (e.g., Ronen, 1983, 1993; Christiansen et al., 2004). There are three types of shipping operations—*industrial operators*, *tramp shippers*, and *liners*. Industrial operators deliver their own cargoes on their own ships at minimal cost, while tramp shippers transport cargoes for other companies. Tramp shippers often have some cargoes under contract that they must ship, *contracts of affreightment*, so general optimization models for industrial and tramp shippers are formulated similarly. Both industrial and tramp shippers often transport additional cargoes from the *spot market* when capacity is available. When an entire ship is available, they will place it on the *spot charter market*, so other shipping companies can charter it. If an industrial or tramp shipper has a cargo that it cannot transport, this cargo is placed on the spot market. The importance of *spot rate costs* is addressed in an example of Fisher and Rosenwein (1989), *the Tanker Division of the Military Sealift Command of the U.S. Navy*, which is responsible for transporting bulk petroleum products world-wide with a fleet of approximately 20 tanker ships. They calculate total profit

by the total spot rate costs that would have incurred if the cargo had been delivered by spot charters minus the operating costs of ships in their fleet. Unlike industrial and tramp shippers, liners operate according to published schedules, so they differ significantly from the other two types of shipping operations. In this paper, we focus on the industrial and tramp-shipper problem.

A *cargo* is the entire content of a ship transported between two ports, and a *schedule* is a sequence of cargoes delivered by the same ship. Ship-scheduling problems are solved by generating a set of feasible delivery schedules for each ship and optimizing a set-packing (or set-partitioning) problem (e.g., Perakis and Bremer, 1992; Kim and Lee, 1997; Bausch et al., 1998; Fagerholt and Christiansen, 2000a,b; Fagerholt, 2001; Christiansen and Fagerholt, 2002). However, traditional set-packing (or set-partitioning) models ignore variability in the spot market.

Each year the *International Tanker Nominal Freight Scale Association Ltd. (ITNFSAL)* calculates a set of values that estimate the cost of shipping between any combination of ports using a standard ship, called *Worldscale (WS) 100*. In addition to these values, *Worldscale* publishes the current market value of shipping freight in terms of a direct percentage of the WS 100 rates (Worldscale, 2000). The fluctuation of WS, or *spot tanker freight rates*, for the past five years is well depicted in Figure 1.

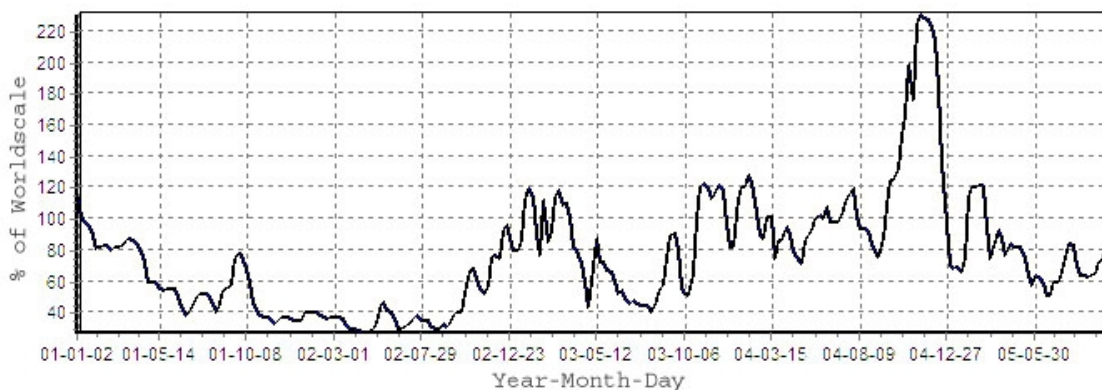


Figure 1: Spot Rates (VLCC-AG/WEST), Weekly: Jan.2001-Oct.2005 (Hanbada Corporation, 2003)

As shown in the figure, the WS is highly variable, and, in OPEC data (OPEC, 2005), the fluctuation rate is as high as 116% of WS 100 within one month. When we convert WS into U.S. dollars, the maximum monthly fluctuation is \$38.47/tonne, which is recorded in Gulf/West route during November and December, 2004. Considering the fact that a Very Large Crude Carrier (VLCC) is in the range of 150,000 to 300,000 tonne, the shipping cost increment of a VLCC could be \$5,770,500 to \$11,541,000 within a month. Even a small change of WS could easily increase shipping costs tens of thousands of dollars. Consequently, the cost of shipping on the spot market is extremely volatile, so managing these fluctuations is critically important for a shipper's success. In this paper, we limit volatility in ship scheduling by constraining the variance of

shipping profit.

2 Formulation

Industrial and tramp shippers must transport contracted cargoes from origin to destination. In addition, they may rent out some of their ships, or they may deliver additional cargoes from the spot market when capacity is available. Conversely, they may also ship some contracted cargoes in the spot market. Most industrial and tramp-shipping problems are modeled as set-packing (or set-partitioning) problems (Christiansen et al., 2004), but we consider the variability of profit. The operating costs of the fleet are relatively constant and controllable compared with the randomness of the spot rate costs, so we focus on the volatility of the spot market.

Let V be the set of ships to be scheduled, and let K be the set of cargoes. Suppose set K is divided into two sets of cargoes: K_1 is the set of cargoes in contracts of affreightment, and K_2 is the set of optionally shipped cargoes from the spot market. For each ship $v \in V$, let F_v denote a set of candidate schedules, and let random variable \tilde{g}_v be the spot rate cost if the company charters out ship v . Let F be the set of all candidate schedules $F = \bigcup_{v \in V} F_v$. For each ship $v \in V$ and each schedule $f \in F_v$, let constant c_{vf} be the cost of covering schedule f with ship v , and let the binary variable

$$x_{vf} = \begin{cases} 1, & \text{if ship } v \text{ covers schedule } f; \\ 0, & \text{otherwise.} \end{cases}$$

For each ship $v \in V$, each schedule $f \in F_v$, and each cargo $k \in K$, the binary constant a_{kvf} indicates whether ship v delivers cargo k in schedule f . For each cargo $k \in K$, let random variables \tilde{r}_k and \tilde{e}_k be the revenue and the spot rate cost of delivering cargo k with a spot charter, respectively.

Then the industrial and tramp shippers ship-scheduling problem can be written as

$$\max \sum_{k \in K_1 \cup K_2} \sum_{v \in V} \sum_{f \in F_v} E[\tilde{r}_k] a_{kvf} x_{vf} - \sum_{k \in K_1} E[\tilde{e}_k] s_k + \sum_{v \in V} E[\tilde{g}_v] u_v - \sum_{v \in V} \sum_{f \in F_v} c_{vf} x_{vf} \quad (1)$$

$$\text{s.t.} \sum_{v \in V} \sum_{f \in F_v} a_{kvf} x_{vf} + s_k = 1, \quad \forall k \in K_1, \quad (2)$$

$$\sum_{v \in V} \sum_{f \in F_v} a_{kvf} x_{vf} \leq 1, \quad \forall k \in K_2, \quad (3)$$

$$\sum_{f \in F_v} x_{vf} + u_v = 1, \quad \forall v \in V, \quad (4)$$

$$x_{vf} \in \{0, 1\}, \quad \forall f \in F_v, v \in V, \quad (5)$$

$$s_k \geq 0, \quad (6)$$

$$u_k \geq 0, \quad (7)$$

where s_k is a binary variable that is equal to one if cargo k is serviced by a spot charter and zero otherwise, and u_v is a binary variable that is equal to one if ship v is chartered out on the spot market and zero otherwise. Variables s_k and u_v need not be defined as binary variables because of constraints (2) and (4). The profit of assigning ship v to cover schedule f is thus given by

$$\sum_{k \in K} \tilde{r}_k a_{kvf} + \sum_{k \in K_1} \tilde{e}_k a_{kvf} - \tilde{g}_v - c_{vf}.$$

Observe that \tilde{e}_k represents the reduction in opportunity cost of having to ship cargo k on the spot market, and similarly \tilde{g}_v is the lost opportunity from using ship v instead of selling it on the spot market.

To simplify notation, let

$$\tilde{r}_{vf} = \sum_{k \in K} \tilde{r}_k a_{kvf} \quad \tilde{e}_{vf} = \sum_{k \in K_1} \tilde{e}_k a_{kvf}.$$

The ship-scheduling problem with limited profit variance (SPLPV) then becomes

$$\max \sum_{v \in V} \sum_{f \in F_v} (E[\tilde{r}_{vf}] + E[\tilde{e}_{vf}] - c_{vf} - E[\tilde{g}_v]) x_{vf} \quad (8)$$

$$\text{s.t.} \sum_{v \in V} \sum_{f \in F_v} a_{kvf} x_{vf} \leq 1, \quad \forall k \in K, \quad (9)$$

$$\sum_{f \in F_v} x_{vf} \leq 1, \quad \forall v \in V, \quad (10)$$

$$\text{var} \left(\sum_{v \in V} \sum_{f \in F_v} (\tilde{r}_{vf} + \tilde{e}_{vf} - c_{vf} - \tilde{g}_v) x_{vf} \right) \leq d, \quad (11)$$

$$x_{vf} \in \{0, 1\}, \quad \forall f \in F_v, v \in V. \quad (12)$$

Here, objective function (8) maximizes expected profit. The constraints in set (9) ensure that cargoes in contracts of affreightment and profitable spot cargoes are serviced, while constraint set (10) implies that each ship in the fleet is assigned to exactly one schedule or chartered out on the spot market. Constraint (11) limits the variance of the profit to a fixed value $d > 0$, a measure traditional ship-scheduling models ignore. Finally (12) represents the binary requirements on the variables.

We can rewrite constraint (11) as

$$\sum_{v_1 \in V} \sum_{f_1 \in F_{v_1}} \sum_{v_2 \in V} \sum_{f_2 \in F_{v_2}} cov(\tilde{r}_{v_1 f_1} + \tilde{e}_{v_1 f_1} - \tilde{g}_{v_1}, \tilde{r}_{v_2 f_2} + \tilde{e}_{v_2 f_2} - \tilde{g}_{v_2}) x_{v_1 f_1} x_{v_2 f_2} \leq d. \quad (13)$$

Variance is non-negative, so the covariance matrix must be symmetric and positive definite (Wu, 2002). For each pair of ships $(v_1, v_2) \in V \times V$, we denote the covariance of costs from assigning ship v_1 to schedule $f_1 \in F_{v_1}$ and assigning ship v_2 to schedule $f_2 \in F_{v_2}$ as $q_{v_1 f_1 v_2 f_2}$. Because the covariance matrix $Q = [q_{v_1 f_1 v_2 f_2}]$ is symmetric and positive definite, the quadratic function $x^T Q x$ is convex. By Kelley's cutting plane method (Kelley, 1960), we can replace the quadratic constraint (13) by an infinite set of first-order constraints given by

$$2 \sum_{v_1 \in V} \sum_{f_1 \in F_{v_1}} \sum_{v_2 \in V} \sum_{f_2 \in F_{v_2}} q_{v_1 f_1 v_2 f_2} w_{v_1 f_1} x_{v_2 f_2} \leq d + \sum_{v_1 \in V} \sum_{f_1 \in F_{v_1}} \sum_{v_2 \in V} \sum_{f_2 \in F_{v_2}} q_{v_1 f_1 v_2 f_2} w_{v_1 f_1} w_{v_2 f_2} \quad \forall w \in \mathfrak{R}^{|F|}. \quad (14)$$

The formulations represented by (8)–(12) and (8)–(10), (12), and (14) are known to be equivalent in convex programming (Kelley, 1960).

2.1 Modeling Random Profits

Market shortages and surpluses may cause large increases and decreases on all chartering rates. Consequently, the spot rate cost \tilde{e}_k will be given by

$$\tilde{e}_k = \alpha_k^e + \beta_k^e \tilde{M} + \tilde{\gamma}_k^e,$$

where \tilde{M} is an independent random variable representing the fluctuation of spot market prices, α_k^e is the expected cost of chartering a ship on the spot market, β_k^e is a constant rate for how the market random variable \tilde{M} changes the spot rate, and $\tilde{\gamma}_k^e$ is an independent random variable for the fluctuation from α_k^e . An implicit assumption is that shipping rates are linearly related to a single market chartering. Because of the dominance of WS on shipping rates, this assumption is reasonable. This type of random profit modeling can often be found in calculating the return on a portfolio (Sharpe, 1970).

The values α_k^e , β_k^e , and $\tilde{\gamma}_k^e$ can be adjusted so that, without loss of generality, $E[\tilde{M}] = E[\tilde{\gamma}_k^e] = 0$, $E[\tilde{e}_k] = \alpha_k^e$, $var(\tilde{M}) = 1$, and $var(\tilde{e}_k) = \beta_k^{e2} + var[\tilde{\gamma}_k^e]$. For each cargo not under contract $k \in K_2$, we let

$\alpha_k^e = \beta_k^e = \tilde{\gamma}_k^e = 0$. The random variables \tilde{r}_k and \tilde{g}_v are analogously defined; that is,

$$\begin{aligned}\tilde{r}_k &= \alpha_k^r + \beta_k^r \tilde{M} + \tilde{\gamma}_k^r, \\ \tilde{g}_v &= \alpha_v^g + \beta_v^g \tilde{M} + \tilde{\gamma}_v^g.\end{aligned}$$

To simplify notation, let $\alpha_k = \alpha_k^r + \alpha_k^e$, $\beta_k = \beta_k^r + \beta_k^e$, $\tilde{\gamma}_k = \tilde{\gamma}_k^r + \tilde{\gamma}_k^e$, and $\beta_{vf} = \sum_{k \in K} \beta_k a_{kvf} + \beta_v^g$. For each pair of ships $(v_1, v_2) \in V \times V$, the covariance of costs for assigning ship v_1 to schedule f_1 and assigning ship v_2 to schedule f_2 , $q_{v_1 f_1 v_2 f_2}$, is given by

$$q_{v_1 f_1 v_2 f_2} = \beta_{v_1 f_1} \beta_{v_2 f_2} + \sum_{k \in f_1 \cap f_2} \text{var}(\tilde{\gamma}_k) + \text{var}(\tilde{\gamma}_{v_1}^g) I_{v_1=v_2}, \quad (15)$$

where the binary constant $I_{v_1=v_2}$ is defined as

$$I_{v_1=v_2} = \begin{cases} 1, & \text{if ships } v_1 \text{ and } v_2 \text{ are the same ship;} \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Tightening Constraints

Each constraint in set (14) can be tightened to

$$2w^T Qx \leq 2\sqrt{dw^T Qw} \quad \forall w \in \mathfrak{R}^{|F|}. \quad (16)$$

By the triangle inequality, for each real vector $w \in \mathfrak{R}^{|F|}$,

$$2\sqrt{dw^T Qw} \leq d + w^T Qw,$$

so constraints in set (16) are at least as tight as those in set (11).

Proposition 1. *For all $w \in \mathfrak{R}^{|F|}$, the associated constraint in set (16) is a valid inequality.*

Proof. The constraints in both (14) and (16) for which $w = 0$ are redundant. For a vector $w \in \mathfrak{R}^{|F|} \setminus \{0^{|F|}\}$, $w^T Qw > 0$ because Q is a positive definite matrix. Let $u = hw$, where h is a positive constant equal to $\sqrt{\frac{d}{w^T Qw}}$. The constraint

$$2u^T Qx \leq d + u^T Qu$$

in the set (14) is a valid inequality. This implies

$$\begin{aligned}2w^T Qx &\leq \frac{d + h^2 w^T Qw}{h} \\ \implies 2w^T Qx &\leq \frac{\sqrt{w^T Qw}}{\sqrt{d}} d + \frac{\sqrt{d}}{\sqrt{w^T Qw}} w^T Qw \\ \implies 2w^T Qx &\leq 2\sqrt{dw^T Qw}.\end{aligned}$$

■

Note that the constraints in set (16) are tangent to the quadratic constraint (11).

3 Branch-and-Price-and-Cut

In Section 2, we described two formulations for ship-scheduling with constrained risk; one used a single quadratic constraint, while the other included an infinite set of first-order constraints. In this section, we develop a branch-and-price-and-cut to solve the latter model.

3.1 Delayed Column-and-Cut Generation

The simplest method to solve the continuous relaxation of SPLPV (CSPLPV) is Enumerated Kelley's Cutting Plane algorithm (EKCP), which is summarized in Algorithm 1.

Algorithm 1 Enumerated Kelley's Cutting Plane Algorithm (EKCP)

Restricted Master Problem (RMP) Step: Let $\mathcal{W} \subset \mathbb{R}^{|F|}$ be a finite set, and solve the linear programming relaxation of (8)–(10), (12), and a subset of constraints (16) using \mathcal{W} to obtain x^* .

if $x^{*T} Q x^* > d + \varepsilon$, where $\varepsilon > 0$ is a very small constant **then**

Cut Generation Step: $\mathcal{W} \leftarrow \mathcal{W} \cup \{x^*\}$ and return to the RMP Step.

else

Return the optimal solution x^* .

end if

In the remainder of this section, we develop a new delayed column-and-cut algorithm which combines delayed column generation with EKCP to solve CSPLPV. Let π and ρ be dual vectors for constraint sets (9) and (10), and (16), respectively. For any optimal solution (x^*, π^*, ρ^*) of CSPLPV, the reduced cost \bar{c}_{vf} of each variable x_{vf} is non-positive; that is

$$\bar{c}_{vf} = E[\tilde{r}_{vf}] + E[\tilde{e}_{vf}] - c_{vf} - E[\tilde{g}_v] - \sum_{k \in K} a_{kvf} \pi_k^* - \pi_v^* - 2 \sum_{w \in \mathcal{W}} \sum_{\tilde{v} \in V} \sum_{\tilde{f} \in F_{\tilde{v}}} \rho_w^* q_{v\tilde{v}\tilde{f}} w_{\tilde{f}} \leq 0, \forall f \in F_v, v \in V. \quad (17)$$

Consider the delayed column-and-cut generation algorithm (DCCG), represented by Algorithm 2, for solving CSPLPV. For the column generation step, we use a topological sorting algorithm to find a new ship schedule with maximum reduced cost (17) on a directed acyclic graph described in Section 3.2.

Traditional EKCP assumes all columns are available, and it adds first-order linear constraints in place of convex constraints as in the cut generation step of DCCG. For large-scale problems, however, enumerating all of the ship schedules is often impractical. Even for medium sized SPLPV problems we need to generate the covariance matrix Q which increases quadratically as the number of columns increases. The computations for constructing such problems grow exponentially, as noted in Section 4.1. Consequently, the use of DCCG is inevitable.

Algorithm 2 Delayed Column-and-Cut Generation Algorithm (DCCG)

Let $\mathcal{W} \leftarrow \emptyset$ be a subset of linear constraints from (16). Generate a subset of ship schedules $\overline{F}_v \subset F_v$, $\forall v \in V$.

RMP Step: Solve CSPLPV over the set of subsets $\overline{F} = \bigcup_{v \in V} \overline{F}_v$ and first-order constraint set \mathcal{W} to get a solution (x^*, π^*, ρ^*) .

if $x^{*T} Q x^* > d + \varepsilon$ **then**

Cut Generation Step: Update the constraint set $\mathcal{W} \leftarrow \mathcal{W} \cup \{x^*\}$ and return to the RMP Step.

else

 Find a ship v and a ship schedule $\overline{f} \in F_v \setminus \overline{F}_v$ that maximizes the *reduced cost* $\overline{c}_{v\overline{f}}$ from (17).

if $\overline{c}_{v\overline{f}} \leq 0$ **then**

 Return the optimal solution (x^*, π^*, ρ^*) .

else

Column Generation Step: $\overline{F} \leftarrow \overline{F} \cup \{\overline{f}\}$ and return to the RMP step.

end if

end if

3.2 Simplified Reduced Cost

For each ship $v \in V$, each schedule $f \in F_v$, and a subset of schedules \overline{F}_v , the reduced cost from (17) is also given by:

$$\begin{aligned} \overline{c}_{vf} = & \sum_{k \in K} \left(\alpha_k - \pi_k^* - 2 \sum_{w \in \mathcal{W}} \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_k \beta_{\hat{v}\hat{f}} + a_{k\hat{v}\hat{f}} \text{var}(\gamma_k) \right) \rho_w^* w_{\hat{v}\hat{f}} \right) a_{kvf} - c_{vf} \\ & + \alpha_v^g - \pi_v^* - 2 \sum_{w \in \mathcal{W}} \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_v^g \beta_{\hat{v}\hat{f}} + I_{v=\hat{v}} \text{var}(\gamma_v^g) \right) \rho_w^* w_{\hat{v}\hat{f}}. \end{aligned} \quad (18)$$

The operating cost c_{vf} is a linear function of the ship v and each consecutive pair of cargo deliveries in the schedule f . Consequently, we can generate a directed network for each ship v , similar to the one in Kim and Lee (1997), to find the schedule f with maximum reduced cost. Each node in the network represents the transportation of each cargo, and each arc represents a consecutive pair of cargo deliveries. The cost of an arc includes the operating cost component of c_{vf} minus the coefficient of a_{kvf} in (18) for the cargo k at the head of the arc. Using this network, we can find a shortest path for each ship $v \in V$ and subtract the lower term in (18) to find the schedule with maximum reduced cost.

The coefficient for a constraint in (16) is given by

$$\begin{aligned}
2a_{wx_vf} = & 2 \sum_{k \in K} \left(\sum_{w \in \mathcal{W}} \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_k \beta_{\hat{v}\hat{f}} + a_{k\hat{f}\hat{v}} \text{var}(\gamma_k) \right) w_{\hat{v}\hat{f}} \right) a_{kvf} \\
& + 2 \sum_{w \in \mathcal{W}} \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_v^g \beta_{\hat{v}\hat{f}} + I_{v=\hat{v}} \text{var}(\gamma_v^g) \right) w_{\hat{v}\hat{f}}.
\end{aligned} \tag{19}$$

Suppose

$$a_{wk} = \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_k \beta_{\hat{v}\hat{f}} + a_{k\hat{f}\hat{v}} \text{var}(\gamma_k) \right) w_{\hat{v}\hat{f}}, \tag{20}$$

$$a_{wv} = \sum_{\hat{v} \in V} \sum_{\hat{f} \in \overline{F}_{\hat{v}}} \left(\beta_v^g \beta_{\hat{v}\hat{f}} + I_{v=\hat{v}} \text{var}(\gamma_v^g) \right) w_{\hat{v}\hat{f}}, \tag{21}$$

then the coefficient is simplified to

$$2a_{wx_vf} = 2a_{wv} + 2 \sum_{k \in f} a_{wf}. \tag{22}$$

The reduced cost simplifies to the following:

$$\bar{c}_{vf} = \sum_{k \in K} \left(\alpha_k - \pi_k^* - \sum_{w \in \mathcal{W}} a_{wk} \rho_w^* \right) a_{kvf} + \alpha_v^g - \pi_v^* - \sum_{w \in \mathcal{W}} a_{wv} \rho_w^* - c_{vf}. \tag{23}$$

The arc costs in the network are now the operating cost component of c_{vf} decreased by

$$\alpha_k - \pi_k^* - \sum_{w \in \mathcal{W}} a_{wk} \rho_w^*. \tag{24}$$

Similarly, the reduced cost of using a ship $v \in V$ is given by

$$\alpha_v^g - \pi_v^* - \sum_{w \in \mathcal{W}} a_{wv} \rho_w^*. \tag{25}$$

For implementation purpose, the cut structure may only include coefficients a_{wk} and a_{wv} instead of a_{wx_vf} . These coefficients are easier to manage, because the number of cargoes $|K|$ and ships $|V|$ are fixed, while the number of schedules in the subset \overline{F} varies as DCCG is executed.

3.3 Follow-On Branching

Vance et al. (1997) showed that follow-on branching, which is a variant of Ryan-Foster branching (Ryan and Foster, 1981), improves the computational efficiency of the deterministic airline crew-scheduling problem. Considering their success, we use follow-on branching for ship-scheduling applications. One advantage to using follow-on branching is ease of applying the branching logic to the column-generation subproblem of DCCG. For the ship scheduling network, follow-on branching implies that we fix or delete certain edges representing a connection between two consecutive cargo deliveries or a deployment of a ship to the first cargo in a candidate schedule.

4 Computational Experiments

In this section, we present the computational results on SPLPV instances. For small problems, we tested EKCP. We implemented the first-order constraint set (16) and the DCCG method within a branch-and-bound tree using COIN/BCP (COIN-OR, 2005). CPLEX 9.120 was used as the LP engine to solve the CSPLPV. To generate columns that have the maximum reduced costs we used the topological sorting algorithm on the network described in Section 3.2. We branched on follow-on variables as explained in Section 3.3. Our experiments were conducted on a Dual 3.06-GHz Intel Xeon Workstation.

4.1 Problem Instances

In our computational analysis, we used modified instances of Kim and Lee (1997), which are similar to those in logistics for world-wide crude oil transportation of a major oil company. A set of cargoes and a set of ships are given for the planning period. In addition to the ship and cargo data, a distance matrix is given. There are two sets of cargoes. The first set of cargoes is contracts of affreightment. The second set of cargoes is from the spot market and may not be shipped depending on the schedule feasibility and/or profit. Each cargo is characterized by size, type, loading date, discharging date, loading port, discharging port, and revenue for lifting cargoes. A ship is assumed to carry only one cargo and can visit several ports in the planning period. Some ships may be chartered out if they have no feasible schedule, or they may also ship contracted cargoes in the spot market. Additional ships may be rented from the spot charter market. Each ship is characterized by size, permitted types of cargo, initial open position, initial open date, speed, fuel consumption, and the daily running costs.

Data sets from Kim and Lee (1997) include 96 ports, 30 ships, and 120 cargoes. Of course, we could create many additional combinations of port, ship, and cargo sets. However, we found that small-sized problems were trivial, so we focused only on medium-sized problems. We created SPLPV instances with the combinations of 30 ships and 30, 60, 90, 120 cargoes for the experiments. The number of variables increased exponentially with respect to the number of cargoes, and SPLPV instances have 1,409 variables, 4,561 variables, 414,369 variables, and 849,498 variables, respectively.

Each SPLPV instance was solved without constraint (11), which is equivalent to a traditional ship scheduling problem, and the variance of schedules in the optimal solution (\overline{var}) was calculated. Each instance was divided into six different levels of limited profit variability by setting constraint (11) equal to 5, 10, 15, 20, 25 and 30 percent standard deviation reduction with respect to $\sqrt{\overline{var}}$. We solved SPLPV instances in all levels using both traditional EKCP and DCCG within branch-and-bound trees, and using a ten-hour time limit.

The variability of spot rates was decided by β and γ . We randomly generated β values between 0 and

$\alpha/3$, and γ values between $(0.05\alpha)^2$ and $(0.15\alpha)^2$. To see these values are practical, we calculated the mean squared percent error of OPEC data mentioned in Section 1, using an exponential growth trend and 6 months of prices to predict next months price. The mean squared percent error lies between 40% and 60%, and that is higher than that of our estimations, 17.7%, which suggested that our instances are conservative. Fifteen medium sized ship scheduling problems were constructed from the ship and cargo data sets, and these β and γ values.

To use EKCP, we had to generate all feasible schedules *a priori* and compare them with each other for constructing a covariance matrix. Though our instances are medium-sized problems, time consumed on constructing ship schedules and covariance matrices grew exponentially as the number of variables increased, which is shown in Figure 2. SPLPV instances with 90 and 120 cargoes could not be constructed within the

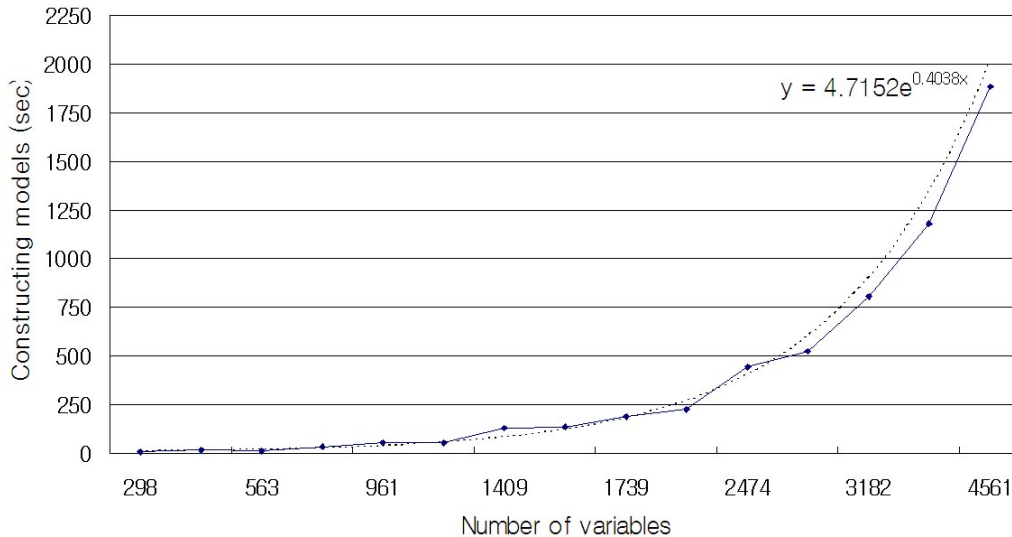


Figure 2: Time spent on constructing models

time limit, which have 414,369 and 849,498 variables, respectively. As a result, we did not use EKCP for these instances. Both EKCP and DCCG can reduce standard deviation to desired levels with reasonable costs, which is shown in Section 4.2.

4.2 Computational Results

Computational results using EKCP are shown in Table 1. The first column values are seven different levels of limited standard deviation d . None of the instances with constrained variance solved to optimality, but in each instance very good solutions were found. The second column shows the standard deviation values of

the best solution found within the time limit, while the third column displays the percentages of standard deviation reduction from \sqrt{var} . The fourth column values are the expected profit of the best solutions found, and the fifth column gives the proportions of the fourth column to the optimal solution found without the quadratic constraint. The column labeled “CPU BS” shows the time spent to the best solution in seconds, and the last column is the number of Kelley cuts generated.

Table 1: Enumerated Kelley’s cutting plane method results

| d | SD | SD (%) | BS | Profit (%) | CPU BS | Cuts |
|--|----------|--------|---------|------------|--------|-------|
| 30 ships, 30 cargoes, 1409 vars x 60 constraints | | | | | | |
| 7434.02 | 7430.57 | 30.03 | 1560313 | 88.51 | 6530 | 88204 |
| 7965.02 | 7933.40 | 25.30 | 1627119 | 92.30 | 6612 | 73456 |
| 8496.02 | 8362.04 | 21.26 | 1665673 | 94.49 | 3568 | 15825 |
| 9027.02 | 8950.39 | 15.72 | 1721997 | 97.68 | 2490 | 9894 |
| 9558.03 | 9435.60 | 11.15 | 1760981 | 99.90 | 2 | 6 |
| 10089.03 | 9962.40 | 6.19 | 1762489 | 99.98 | 2 | 8 |
| 10620.03 | 10620.03 | 0.00 | 1762832 | 100.00 | 0 | 0 |
| 30 ships, 60 cargoes, 4561 vars x 89 constraints | | | | | | |
| 12767.97 | 12742.84 | 30.14 | 2528840 | 85.61 | 1744 | 361 |
| 13679.97 | 13608.75 | 25.39 | 2682384 | 90.81 | 34188 | 7483 |
| 14591.96 | 14454.00 | 20.76 | 2738172 | 92.70 | 14636 | 3315 |
| 15503.96 | 15389.90 | 15.63 | 2826678 | 95.69 | 19010 | 2699 |
| 16415.96 | 16376.45 | 10.22 | 2886632 | 97.72 | 6902 | 746 |
| 17327.96 | 17312.22 | 5.09 | 2920067 | 98.85 | 19629 | 1480 |
| 18239.96 | 18239.96 | 0.00 | 2953939 | 100.00 | 0 | 0 |

With a small expected profit reduction, we can significantly limit standard deviation. For example, with less than 5% profit reduction, we can decrease standard deviation by 15.63%, which is shown in the results with 30 ships and 60 cargoes. The relationship between standard deviation restriction and profit reduction is depicted by the efficient frontiers in Figure 3.

Table 2 presents computational results using DCCG for instances that have 30, 60, 90, and 120 cargoes. The first to seventh columns are the same as those in Table 1. The column labeled “Vars” displays the number of ship schedules generated to the best solution found, and “CPU Vars” shows the time spent generating ship schedules.

DCCG becomes the practical method as the problem size increases, although EKCP performs better than DCCG for small instances. DCCG can also significantly limit standard deviation with a small profit

Table 2: Delayed column-and-cut generation results

| d | SD | SD (%) | BS | Profit (%) | CPU BS | Cuts | Vars | CPU Vars |
|--|----------|--------|---------|------------|--------|--------|--------|----------|
| 30 ships, 30 cargoes, 1409 vars x 60 constraints | | | | | | | | |
| 7434.02 | 7367.90 | 30.62 | 1443665 | 81.89 | 2146 | 209606 | 97652 | 405 |
| 7965.02 | 7942.22 | 25.21 | 1507933 | 85.54 | 720 | 28200 | 27213 | 124 |
| 8496.02 | 8475.97 | 20.19 | 1553957 | 88.15 | 3201 | 431175 | 192270 | 658 |
| 9027.02 | 8981.53 | 15.43 | 1608842 | 91.26 | 76 | 3390 | 1850 | 11 |
| 9558.03 | 9516.31 | 10.39 | 1659654 | 94.15 | 57 | 2118 | 2303 | 10 |
| 10089.03 | 10044.60 | 5.42 | 1698373 | 96.34 | 8984 | 473007 | 509687 | 2079 |
| 10620.03 | 10620.03 | 0.00 | 1762832 | 100.00 | 2 | 0 | 90 | 0 |
| 30 ships, 60 cargoes, 4561 vars x 89 constraints | | | | | | | | |
| 12767.97 | 12728.12 | 30.22 | 2472711 | 83.71 | 21502 | 213467 | 198496 | 5163 |
| 13679.97 | 13679.66 | 25.00 | 2575115 | 87.18 | 28262 | 257952 | 249273 | 7095 |
| 14591.96 | 14551.91 | 20.22 | 2671811 | 90.45 | 61 | 232 | 670 | 18 |
| 15503.96 | 15454.84 | 15.27 | 2756273 | 93.31 | 73 | 151 | 627 | 22 |
| 16415.96 | 16383.86 | 10.18 | 2830737 | 95.83 | 31452 | 207321 | 294414 | 9052 |
| 17327.96 | 17326.48 | 5.01 | 2894907 | 98.00 | 22033 | 93115 | 290519 | 7426 |
| 18239.96 | 18239.96 | 0.00 | 2953940 | 100.00 | 11 | 0 | 190 | 3 |
| 30 ships, 90 cargoes, 414369 vars x 119 constraints | | | | | | | | |
| 22678.81 | 22667.33 | 30.04 | 4155973 | 81.15 | 35392 | 16541 | 17307 | 21182 |
| 24298.73 | 24235.86 | 25.19 | 4331288 | 84.58 | 5403 | 2597 | 3035 | 3794 |
| 25918.64 | 25892.53 | 20.08 | 4498549 | 87.84 | 33254 | 14105 | 19828 | 20356 |
| 27538.56 | 27534.32 | 15.01 | 4665335 | 91.10 | 24544 | 3764 | 15939 | 18780 |
| 29158.47 | 29149.92 | 10.03 | 4803991 | 93.81 | 35625 | 16975 | 19904 | 20004 |
| 30778.39 | 30664.88 | 5.35 | 4958686 | 96.83 | 4872 | 683 | 3009 | 3604 |
| 32398.30 | 32398.30 | 0.00 | 5121210 | 100.00 | 1059 | 0 | 550 | 1045 |
| 30 ships, 120 cargoes, 849498 vars x 148 constraints | | | | | | | | |
| 24989.61 | 24982.95 | 30.02 | 5116365 | 85.79 | 18805 | 3499 | 4661 | 14942 |
| 26774.58 | 26674.91 | 25.28 | 5315988 | 89.14 | 34197 | 5415 | 8262 | 27085 |
| 28559.55 | 28556.75 | 20.01 | 5489632 | 92.05 | 17923 | 2630 | 3926 | 15098 |
| 30344.52 | 30315.64 | 15.08 | 5591225 | 93.76 | 22301 | 1451 | 4919 | 18051 |
| 32129.50 | 32058.70 | 10.20 | 5748335 | 96.39 | 16705 | 1328 | 3975 | 14307 |
| 33914.47 | 33885.69 | 5.08 | 5847119 | 98.05 | 19288 | 703 | 4224 | 16643 |
| 35699.44 | 35699.44 | 0.00 | 5963505 | 100.00 | 3891 | 0 | 686 | 3735 |

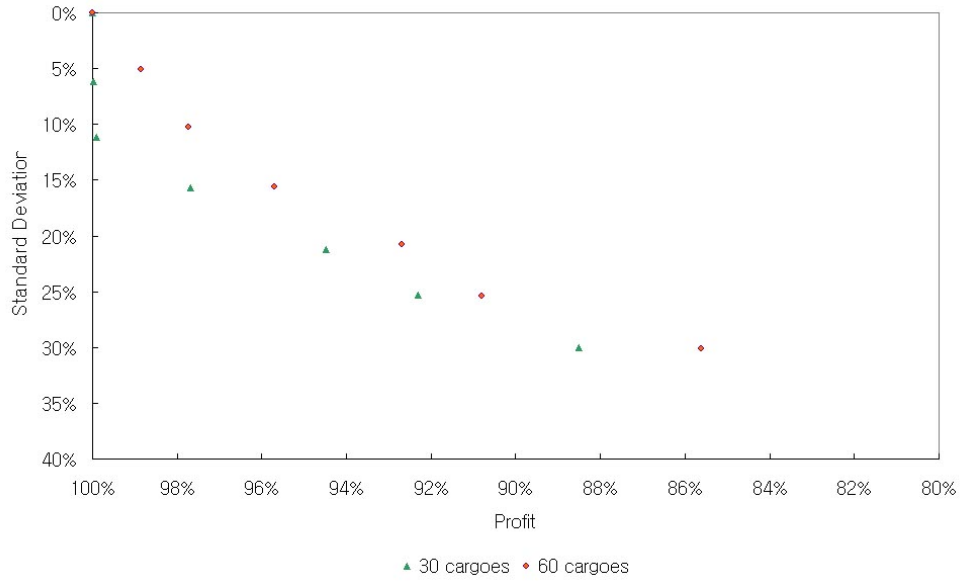


Figure 3: Enumerated Kelley’s cutting plane method

reduction. For example, with only 4.17% profit reduction, we can restrict the deviation to 10.18% as shown in the instance with 30 ships and 60 cargoes instance. The efficient frontiers are depicted in Figure 4.

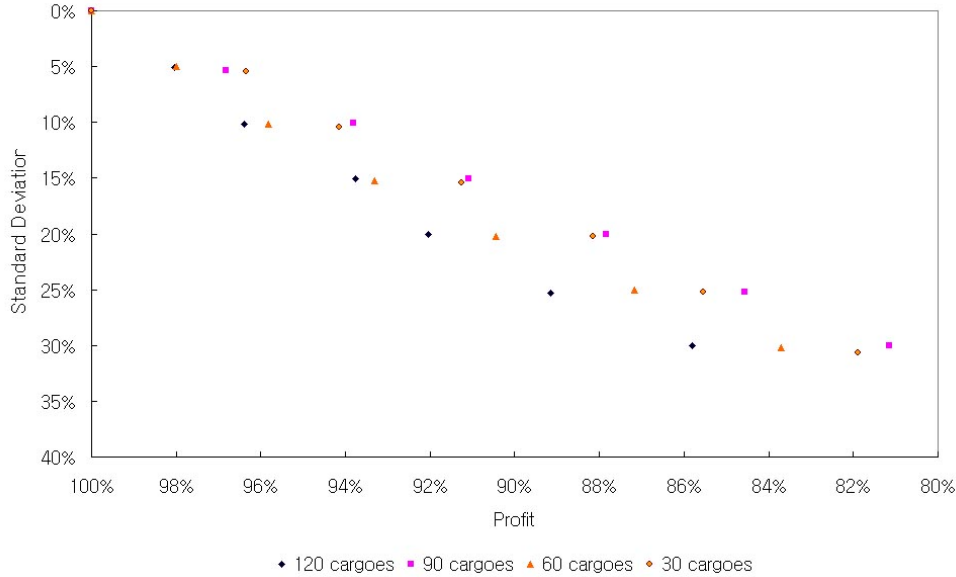


Figure 4: Delayed column and cut generation

5 Conclusions and Future Research

As economic fluctuation increases, logistics planners must find methods of managing risk. In this paper, we presented a new set-packing model for ship scheduling problems, which has a quadratic variance constraint and limits the risk of the fluctuation in the spot market. We used traditional Kelley’s cutting plane algorithm and a delayed column and cut generation algorithm (DCCG) on medium-sized ship-scheduling problems with restricted variance. To use Kelley’s cutting plane algorithm, we enumerated all feasible schedules *a priori* with the covariance matrix. As the number of schedules increased, time for constructing instances grew exponentially and using Kelley’s cutting plane algorithm became impractical, a fact that motivated us to develop DCCG. In each iteration of DCCG, we add either a new Kelley’s cut or a new schedule with maximum reduced cost. The new schedule is found by using a topological sorting algorithm on a directed acyclic graph. Computational experiments with instances similar to those in logistics for world-wide crude oil transportation of a major oil company showed that both Kelley’s cutting plane algorithm and DCCG can reduce variance significantly with reasonable expected profit reduction. Even though neither method could optimize medium-sized instances within a ten-hour time limit, very good solutions were found. Because adding multiple cuts and columns in each iteration of branch-and-price-and-cut often improves computational efficiency, developing methods to add multiple cuts and columns in each iteration of DCCG is a topic of future research.

6 Acknowledgments

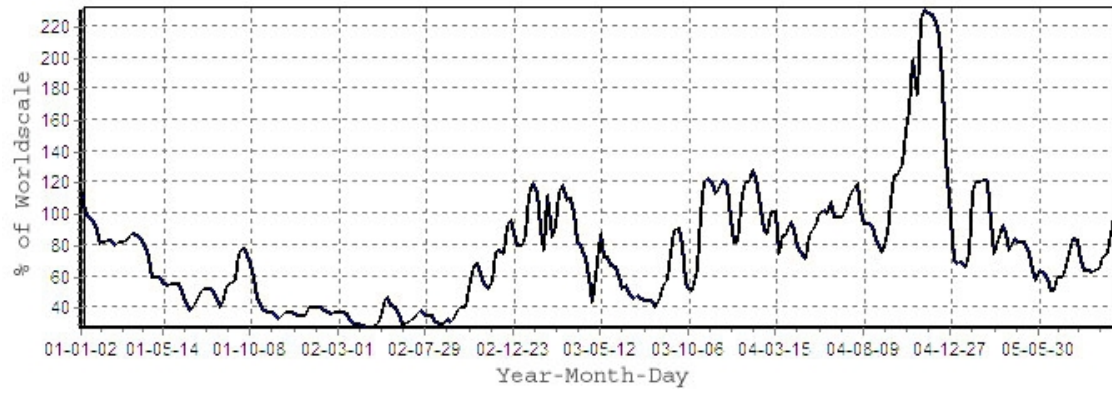
We would like to thank Professor Si-Hwa Kim, a visiting scholar from Korea Maritime University, Busan, Korea, for providing example ship-scheduling problems. This research is partially sponsored by Texas Advanced Technology Program Grant No. 003656-0197-2003.

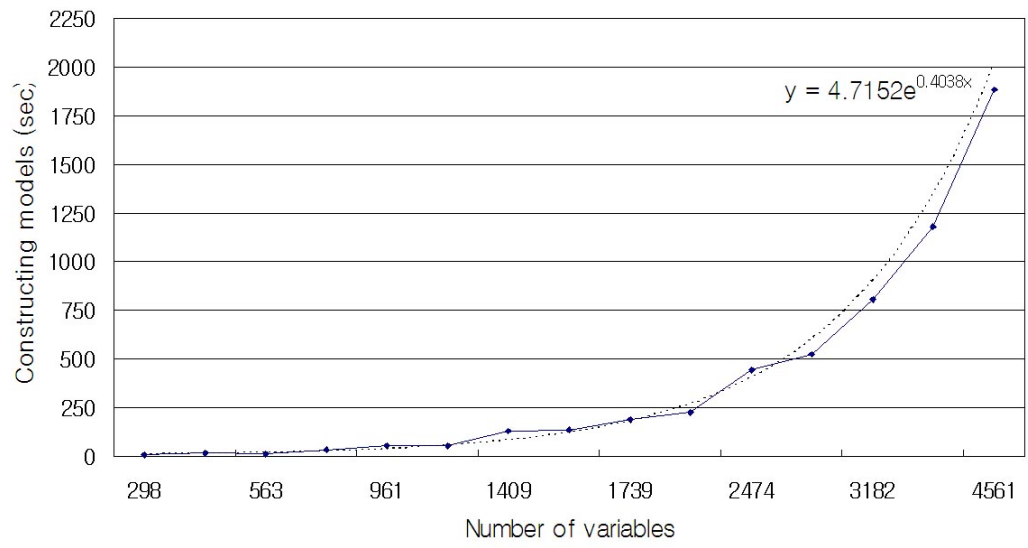
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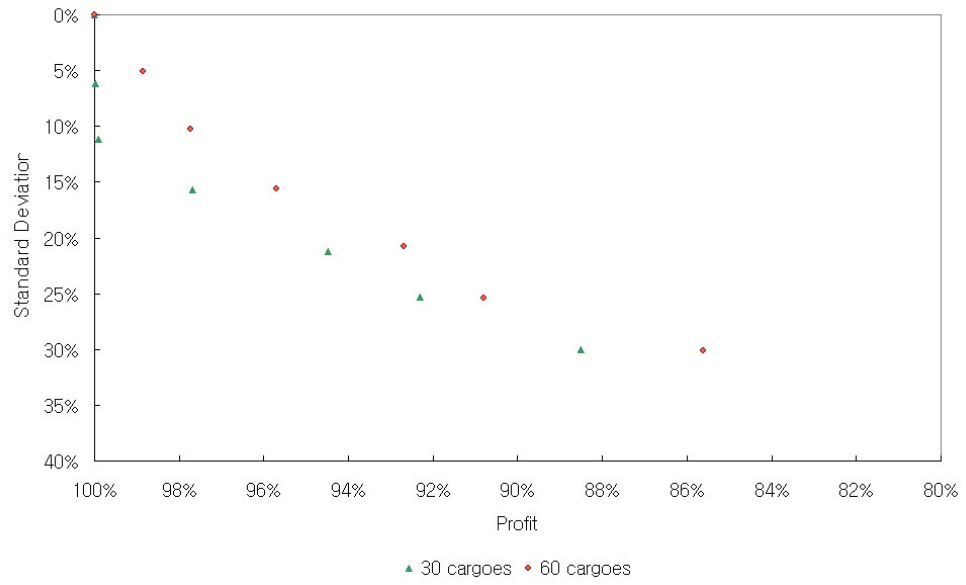
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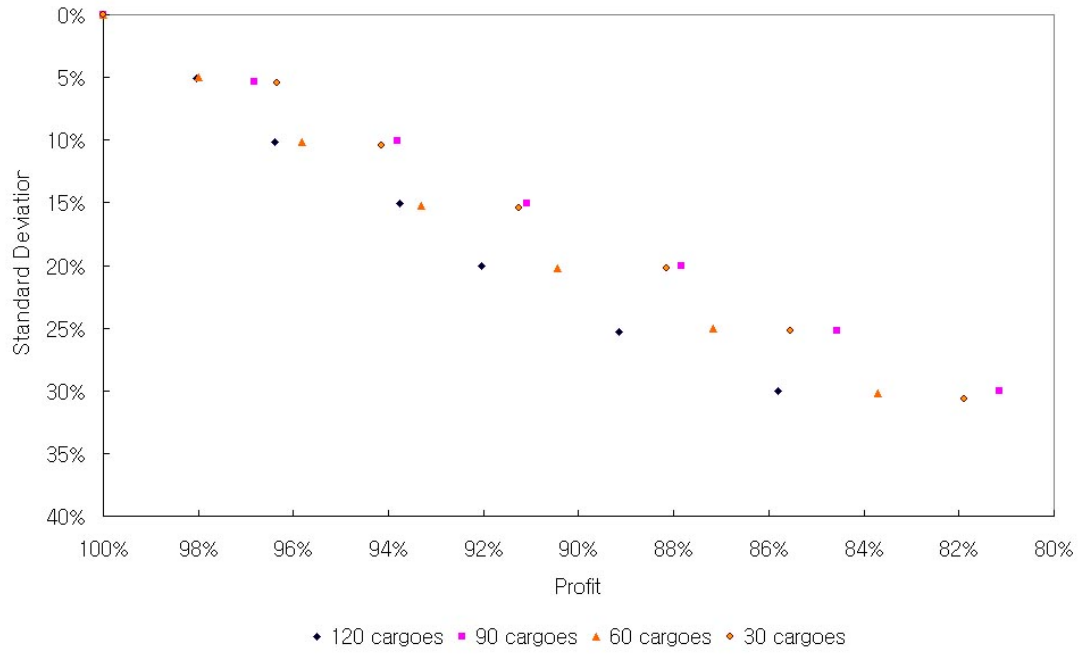
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| d | SD | SD (%) | BS | Profit (%) | CPU BS | Cuts |
|--|----------|--------|---------|------------|--------|-------|
| 30 ships, 30 cargoes, 1409 vars x 60 constraints | | | | | | |
| 7434.02 | 7430.57 | 30.03 | 1560313 | 88.51 | 6530 | 88204 |
| 7965.02 | 7933.40 | 25.30 | 1627119 | 92.30 | 6612 | 73456 |
| 8496.02 | 8362.04 | 21.26 | 1665673 | 94.49 | 3568 | 15825 |
| 9027.02 | 8950.39 | 15.72 | 1721997 | 97.68 | 2490 | 9894 |
| 9558.03 | 9435.60 | 11.15 | 1760981 | 99.90 | 2 | 6 |
| 10089.03 | 9962.40 | 6.19 | 1762489 | 99.98 | 2 | 8 |
| 10620.03 | 10620.03 | 0.00 | 1762832 | 100.00 | 0 | 0 |
| 30 ships, 60 cargoes, 4561 vars x 89 constraints | | | | | | |
| 12767.97 | 12742.84 | 30.14 | 2528840 | 85.61 | 1744 | 361 |
| 13679.97 | 13608.75 | 25.39 | 2682384 | 90.81 | 34188 | 7483 |
| 14591.96 | 14454.00 | 20.76 | 2738172 | 92.70 | 14636 | 3315 |
| 15503.96 | 15389.90 | 15.63 | 2826678 | 95.69 | 19010 | 2699 |
| 16415.96 | 16376.45 | 10.22 | 2886632 | 97.72 | 6902 | 746 |
| 17327.96 | 17312.22 | 5.09 | 2920067 | 98.85 | 19629 | 1480 |
| 18239.96 | 18239.96 | 0.00 | 2953939 | 100.00 | 0 | 0 |

| d | SD | SD (%) | BS | Profit (%) | CPU BS | Cuts | Vars | CPU Vars |
|--|----------|--------|---------|------------|--------|--------|--------|----------|
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| 7965.02 | 7942.22 | 25.21 | 1507933 | 85.54 | 720 | 28200 | 27213 | 124 |
| 8496.02 | 8475.97 | 20.19 | 1553957 | 88.15 | 3201 | 431175 | 192270 | 658 |
| 9027.02 | 8981.53 | 15.43 | 1608842 | 91.26 | 76 | 3390 | 1850 | 11 |
| 9558.03 | 9516.31 | 10.39 | 1659654 | 94.15 | 57 | 2118 | 2303 | 10 |
| 10089.03 | 10044.60 | 5.42 | 1698373 | 96.34 | 8984 | 473007 | 509687 | 2079 |
| 10620.03 | 10620.03 | 0.00 | 1762832 | 100.00 | 2 | 0 | 90 | 0 |
| 30 ships, 60 cargoes, 4561 vars x 89 constraints | | | | | | | | |
| 12767.97 | 12728.12 | 30.22 | 2472711 | 83.71 | 21502 | 213467 | 198496 | 5163 |
| 13679.97 | 13679.66 | 25.00 | 2575115 | 87.18 | 28262 | 257952 | 249273 | 7095 |
| 14591.96 | 14551.91 | 20.22 | 2671811 | 90.45 | 61 | 232 | 670 | 18 |
| 15503.96 | 15454.84 | 15.27 | 2756273 | 93.31 | 73 | 151 | 627 | 22 |
| 16415.96 | 16383.86 | 10.18 | 2830737 | 95.83 | 31452 | 207321 | 294414 | 9052 |
| 17327.96 | 17326.48 | 5.01 | 2894907 | 98.00 | 22033 | 93115 | 290519 | 7426 |
| 18239.96 | 18239.96 | 0.00 | 2953940 | 100.00 | 11 | 0 | 190 | 3 |
| 30 ships, 90 cargoes, 414369 vars x 119 constraints | | | | | | | | |
| 22678.81 | 22667.33 | 30.04 | 4155973 | 81.15 | 35392 | 16541 | 17307 | 21182 |
| 24298.73 | 24235.86 | 25.19 | 4331288 | 84.58 | 5403 | 2597 | 3035 | 3794 |
| 25918.64 | 25892.53 | 20.08 | 4498549 | 87.84 | 33254 | 14105 | 19828 | 20356 |
| 27538.56 | 27534.32 | 15.01 | 4665335 | 91.10 | 24544 | 3764 | 15939 | 18780 |
| 29158.47 | 29149.92 | 10.03 | 4803991 | 93.81 | 35625 | 16975 | 19904 | 20004 |
| 30778.39 | 30664.88 | 5.35 | 4958686 | 96.83 | 4872 | 683 | 3009 | 3604 |
| 32398.30 | 32398.30 | 0.00 | 5121210 | 100.00 | 1059 | 0 | 550 | 1045 |
| 30 ships, 120 cargoes, 849498 vars x 148 constraints | | | | | | | | |
| 24989.61 | 24982.95 | 30.02 | 5116365 | 85.79 | 18805 | 3499 | 4661 | 14942 |
| 26774.58 | 26674.91 | 25.28 | 5315988 | 89.14 | 34197 | 5415 | 8262 | 27085 |
| 28559.55 | 28556.75 | 20.01 | 5489632 | 92.05 | 17923 | 2630 | 3926 | 15098 |
| 30344.52 | 30315.64 | 15.08 | 5591225 | 93.76 | 22301 | 1451 | 4919 | 18051 |
| 32129.50 | 32058.70 | 10.20 | 5748335 | 96.39 | 16705 | 1328 | 3975 | 14307 |
| 33914.47 | 33885.69 | 5.08 | 5847119 | 98.05 | 19288 | 703 | 4224 | 16643 |
| 35699.44 | 35699.44 | 0.00 | 5963505 | 100.00 | 3891 | 0 | 686 | 3735 |

Restricted Master Problem (RMP) Step: Let $\mathcal{W} \subset \mathbb{R}^{|F|}$ be a finite set, and solve the linear programming relaxation of (8)–(10), (12), and a subset of constraints (16) using \mathcal{W} to obtain x^* .

if $x^{*T} Q x^* > d + \varepsilon$, where $\varepsilon > 0$ is a very small constant **then**

Cut Generation Step: $\mathcal{W} \leftarrow \mathcal{W} \cup \{x^*\}$ and return to the RMP Step.

else

 Return the optimal solution x^* .

end if

Let $\mathcal{W} \leftarrow \emptyset$ be a subset of linear constraints from (16). Generate a subset of ship schedules $\bar{S} \subset S$.

RMP Step: Solve CSPLPV over the set of subsets \bar{S} and first-order constraint set \mathcal{W} to get a solution (x^*, π^*, ρ^*) .

if $x^*Qx^* - d > \varepsilon$ **then**

Cut Generation Step: Update the constraint set $\mathcal{W} \leftarrow \mathcal{W} \cup \{x^*\}$ and return to the RMP Step.

end if

Find a ship schedule $\bar{s} \in S \setminus \bar{S}$ that maximizes the *reduced cost* $\bar{c}_{\bar{s}}$ from (17).

if $\bar{c}_{\bar{s}} \leq 0$ **then**

 Return the optimal solution (x^*, π^*, ρ^*) .

else

Column Generation Step: $\bar{S} \leftarrow \bar{S} \cup \{\bar{s}\}$ and return to the RMP step.

end if

Figure Captions

Figure 1: Spot Rates (VLCC-AG/WEST), Weekly: Jan.2001-Oct.2005 (Hanbada Corporation, 2003).

Figure 2: Time spent on constructing models.

Figure 3: Enumerated Kelley's cutting plane method.

Figure 4: Delayed column and cut generation.

Table Captions

Table 1: Enumerated Kelley's cutting plane method results.

Table ?? : Delayed column-and-cut generation results.

Algorithm Captions

Algorithm 1: Enumerated Kelley's Cutting Plane Algorithm.

Algorithm 2: Delayed Column-and-Cut Generation Algorithm.