Nonparametric Multivariate Control Charts Based on A Linkage Ranking Algorithm

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COSMOS Technical Report 09-05

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Abstract

Control charts have been widely recognized as important and critical tools in system monitoring for detection of abnormal behavior and quality improvement. In particular, multivariate control charts have been effectively used when a process involves a number of correlated process variables. Most existing multivariate control charts were developed using the assumption of normally distributed process variables. However, process data from modern industries often do not follow the normal distribution. Despite the great need for nonparametric control charts that can control the error rate regardless of the underlying distribution, few efforts have been made in this direction. In this paper, we propose a new nonparametric control chart (called the *k*LINK chart) based on a *k*-linkage ranking algorithm that calculates the ranking of a new measurement relative to the in-control training data. A simulation study was performed to demonstrate the effectiveness of our *k*LINK chart and its superiority over the traditional Hotelling's T^2 chart and the ranking depth control chart in nonnormal situations. In addition, to enable increased sensitivity to small shifts, we present an exponentially weighted moving average version of a *k*LINK chart.

Key words: nonparametric, multivariate control charts, statistical quality control, Hotelling's T^2, data depth

1. Introduction

Multivariate quality control monitors the quality of a production process that depends on several correlated quality characteristics. A sample quality characteristic is used to calculate measurements, such as sample means, which are then combined into a single statistic that may be plotted on a control chart. The main purpose of the control chart is to monitor the performance of a process over time to maintain the process in-control.

Most of the research performed in this field assumes the measurements that describe quality characteristics follow a multivariate normal distribution[1-3, 12, 18, 20, 27]. A Hotelling's T^2 control chart (T^2 chart)[11] is the most widely used multivariate control chart under the assumption that the measurements follow a multivariate normal distribution. However, this assumption of normality is not always applicable. In particular, the distribution of the process variable may be highly skewed. In practice, use of transformed variables can be suggested. However, this task is difficult, and statistical methods to effectively transform multivariate nonnormal data into multivariate normal data are very limited[22].

When this assumed normality is not present, the calculated probabilities of Type I and Type II error rates derived from the control mechanisms are unreliable. Only a limited number of methods of multivariate quality control are available for use in nonnormal situations, which gives rise to the motivation to develop nonparametric methods for multivariate process control.

Nonparametric techniques control the probabilities of false alarms no matter what the underlying distribution of the quality characteristics is. In the absence of a distributional assumption, a nonparametric procedure requires training the data of the in-control measurements to represent the underlying distribution. Research in nonparametric multivariate quality control has been conducted by Beltran[4], Bush[6], Bush et al.[7], Cheng et al.[9], Hayter and Bush[10], Kapatou and Reynolds[14], Liu[16], Qiu[22], Qiu and Hawkins[23], Stoumbos and Jones[24], and Stoumbos and Reynolds[25]. An overview of some of these methods can be found in Chakraborti et al.[3].

The nonparametric r chart (ranking depth chart) introduced by Liu[16] is based on ranking the depth of the multivariate testing data relative to the multivariate training data and then plotting these ranks in a univariate control chart. Data depth is a measure of how central a data point is compared with a data cloud without any distributional assumption. Several data depths, such as Mahalanobis depth and simplicial depth, have been used to construct r charts[15]. More recently, Beltran[4] extended Liu's ranking depth chart by integrating principal components analysis (PCA). His PCA-simplicial depth r control charts were constructed by using simplicial depth ranks of the principal components to improve the detection of both variability and correlation shifts in multivariate processes without any distributional assumptions.

Recently, Qiu[22] proposed a nonparametric multivariate control chart based on log-linear modeling. He proposed to estimate the in-control distribution of the original training data $\mathbf{X}(i)$ by transforming $\mathbf{X}(i)$ into binary forms $\mathbf{Y}(i)$ to enable a log-linear modeling approach for estimating the joint distribution of $\mathbf{Y}(i)$. Even though information is lost in the transformation, Qiu's method has the capability to detect a shift in a location parameter vector (e.g., the median vector). However, when a large number of variables are involved, constructing log-linear models is a challenge. In such cases, the performance of log-linear models has been inconclusive.

Although all of these nonparametric control charts perform reasonably well in the situations for which they were designed, no consensus exists about which of them best satisfies all conditions. In the absence of consensus, we propose a new ranking algorithm, k-linkage ranking (kLINK), and construction of a kLINK chart. We demonstrate that this is a logical, efficient, and robust control chart with flexible control boundaries that can effectively monitor multivariate processes in nonnormal situations.

The outline of this paper is follows. Section 2 describes the general basis for the *k*LINK nonparametric procedure and presents our *k*LINK control chart. Section 3 provides simulation results that show the effectiveness of *k*LINK nonparametric charts compared with existing multivariate charts, such as the T^2 and ranking depth charts in nonnormal situations. Section 4 presents an exponentially weighted moving average version of a *k*LINK. Our conclusions are presented in Section 5.

2. The kLINK Control Chart

2.1 kLINK nonparametric procedure

Denote the vector of p quality characteristic measurements by $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, where $\mathbf{x}$ has covariance matrix $\mathbf{\Sigma}$. Assume there exists an training dataset of m independent, identically distributed measurements, $\mathbf{x}^1, \ldots \mathbf{x}^m$, where each is a p-dimensional vector of measurements, $\mathbf{x}^i = (x_1^i, \ldots, x_p^i)$ for $i = 1, \ldots, m$. Note that each measurement may be based on a single observation or a calculation based on multiple measurements. The purpose of a quality control procedure is to test the null hypothesis that a new measurement is from the same (unknown) distribution as the training data. If the null hypothesis is not true, a change has occurred within the process that has affected one or more of the quality characteristics, and the process should be declared out of control. When the process is operating at a constant mean, and variability is caused only by unavoidable sources, the process is said to be in control. An out-of-control process is operating under assignable causes of variability. These assignable causes should be detected and eliminated.

In classical quality control, the training data (or Phase I data) are used to calculate control limits. Nonparametric procedures rely on training data to define the standard against which new measurements are compared. When unknown distributions are involved, several procedures have been developed to determine if the training data can be considered in control. These procedures look for outliers and changes over time in the median of the training data[6]. Given appropriately calibrated training data, our *k*LINK nonparametric procedure calculates score statistics that rank the quality characteristic measurements of a new testing measurement, $\mathbf{x}^0$, relative to the measurements in the training data. Specifically, $\mathbf{x}^0$ is added to the training data to form the combined data of $m + 1$ measurements, then all $m + 1$ measurements are then ranked, and the value plotted on the control chart is based on the resulting ranking of $\mathbf{x}^0$.

For a graphical illustration of the *k*LINK nonparametric procedure, consider the dataset, illustrated in Figure 1 (discussed in more detail in Section 4). Our *k*LINK method resulted in the rankings {1, 2,…, 100} that label the measurements. Low rankings indicate measurements that are central to the combined data; the highest rankings indicate those that are at the fringes. In a two-dimensional case, a new measurement can be plotted with the training data, and a visual judgment can be made as to whether or not the new measurement comes from the same distribution. In higher dimensions, however, visualization becomes difficult. However, rankings are simple to interpret regardless of the number of quality characteristics.

[Figure 1 about here]

Scores and ranks are calculated for all measurements in the combined data. A training data measurement $\mathbf{x}^i$ has score S_i and the corresponding ranking R_i (i = 1,…, m). A measurement

$\mathbf{x}_0$ has score S_0 and the corresponding ranking R_0. The rankings are obtained by simply ordering the scores (average ranks can be used if there are ties among the scores). A low rank R_0 indicates that the new measurement is within the borders of the p-dimensional space, which is defined by the training data. If a new measurement and all m measurements in the training data are from the same distribution (so that the process is in control), then R_0 is equally likely to take any value from one to $m + 1$ (assuming there are no ties in the scores). Large values of R_0 indicate that the new measurement is outside the training data. The plausibility that the process is in-control can be calculated as

$$\varphi = \frac{m + 2 - R_0}{m + 1}. \tag{1}$$

The φ represents the proportion of the $m + 1$ measurements in the combined pool that have scores S_i no smaller than S_0. It is important to note that the smallest possible φ obtained from the nonparametric procedures is limited by the size of the training data.

To equalize the effects of quality characteristics with different variances and the relative contribution of correlated quality characteristics, the distance between two measurements in the multivariate space is commonly determined by the Mahalanobis distance between measurements $\mathbf{x}^i$ and $\mathbf{x}^j$ [5, 13]: $D^2 = (\mathbf{x}^i - \mathbf{x}^j)'\Sigma^{-1}(\mathbf{x}^i - \mathbf{x}^j)$. When Σ is unknown, as we assume in this paper, the sample covariance matrix $\mathbf{V}$ is used to calculate the distance:

$$d_{ij} = (\mathbf{x}^i - \mathbf{x}^j)'\mathbf{V}^{-1}(\mathbf{x}^i - \mathbf{x}^j). \tag{2}$$

In the following section, we describe in detail our k-linkage ranking algorithm that is used to create the kLINK nonparametric multivariate quality control chart.

2.2. Linkage ranking algorithm

Our linkage ranking algorithm is based on cluster analysis and chaining. In cluster analysis, we are interested in separating the data into distinct groups. Chaining begins with one starting data point, then measurements are added to the chain one by one to form one large cluster instead of several smaller clusters. Measurements are continually closer or more similar to the cluster than to the other measurements; thus, no separation occurs and dissimilarities are not discovered. Although certain clustering techniques are designed to avoid this problem, the concept of chaining is the key to linkage ranking algorithms. Again, the fundamental question in quality control is whether a training dataset and a new measurement have the same distribution, or in other words, whether they belong to the same cluster. If a new measurement belongs to the cluster defined by the training data, then it will quickly be linked to the rest of the chain.

The chain begins at the center of the distribution, which we represent by the central statistic, and then branches to all $m + 1$ measurements in the combined data. The central statistic $x^M = (x_1^M, ..., x_p^M)$ is intended to represent the middle of the distribution — for example, the sample mean or the sample median of the training data. At each step, the closest measurement to the cluster is added. A measurement's distance to the cluster may be defined in several ways, which will be discussed later. The score statistic S_i is the order in which measurement realization $\mathbf{x}^i$ is linked to the chain:

$$S_i = j \text{ when } \mathbf{x}^i \text{ is the } j^{th} \text{ measurement linked to the chain,}$$

and hence, rank $R_i = S_i$.

The chain will spread outward from the center to the fringes. The linkage ranking algorithm tends to allocate lower scores to measurements in dense areas; this is because once one measurement in a dense area is added to the chain, the others will soon follow. Having no close neighbors, measurements in sparse areas will not be linked to the chain as quickly.

The central statistic is considered the initial member of the chain, although one may argue that once the first measurement from the combined data is linked to the chain, then the central statistic should be dropped because it is not an actual measurement. However, if we drop the central statistic at this point, then the measurement that was second closest to the central statistic may not be the next measurement to join the chain unless it also happens to be the closest measurement to the first one linked. Thus, a measurement's distance to the central statistic should influence a decision about which measurement will join the chain at the next iteration. As the chain grows, the weight of this decision will decrease. By the time the last few measurements are joined to the chain, the decision to keep or discard the central statistic will no longer be influential. Because the control procedure is focused on the measurements made at the fringes, keeping the central statistic may not be critical to the success of the procedure; however, its retention is recommended.

The question that remains is how to define the distance from a measurement to the chain. The distance to the chain is a function of the distances to all the measurements in the chain. Suppose the chain has g measurements; then $m + 1 - g$ measurements remain to be linked to the chain. For a measurement $\mathbf{x}^i$ not in the chain, calculate its Mahalanobis distance to every $\mathbf{x}^j$ in the chain. Let $D_i^{(h)}$ be the h^{th} smallest of these distances, $h = 1,\ldots, g$. Then calculate the distance from measurement $\mathbf{x}^i$ to the chain as

$$T_i = \sum_{h=1}^{k} D_i^{(h)} , \tag{3}$$

where k should be determined by the user. The linking rule is to add the measurement $\mathbf{x}^i$ with the smallest T_i. Thus, a measurement's distance to the chain is the minimum distance from it to any k measurements already in the chain.

Our k-linkage ranking (kLINK) algorithm uses linking via the sum of k distances in Equation (3) to calculate all the ranks R_i, and, subsequently, the rank R_0 for a new testing measurement. Equation (1) is used to calculate the appropriate φ. A summary of the kLINK algorithm is described as follows:

Algorithm *k*LINK

Specify k
Calculate central statistic $\mathbf{x}^M$ of the training data.
Initialize the set of measurements $\mathbf{x}^j$ in the chain: CHAIN = $\{M\}$.
Initialize the set of measurements $\mathbf{x}^i$ not in the chain: NOTinCHAIN = $\{0, 1, \ldots, m\}$.
Initialize counter: RANK = 0.
repeat
    for all measurements $\mathbf{x}^i$ such that $i \in$ NOTinCHAIN **do**
        for all measurements $\mathbf{x}^j$ such that $j \in$ CHAIN **do**
            Calculate d_{ij} as in equation (2).
        end for
        Calculate T_i as in equation (3).
        if T_i is the smallest so far **then**
            Save index i as i^*.
        end if
    end for
    Add $\mathbf{x}^{i^*}$ with smallest T_{i^*} (saved) to CHAIN:
    CHAIN = CHAIN $\cup \{i^*\}$;
    NOTinCHAIN = NOTinCHAIN \ $\cup \{i^*\}$;
    RANK = RANK +1;
    R_i = RANK.
until all measurements are linked in CHAIN.

The method based on linking can be thought of as a nearest-neighbor method. The parameter k determines how many nearest neighbors are considered, and only measurements already in the chain are a nonmember's potential neighbors. When k = 1, the only consideration given to a measurement joining the chain is its distance to any one member of the chain. When $k = g$, the measurement selected to join depends on its distance to every point in the chain. Ideally, the center of the chain should remain close to the center of the training data. When k is large, the center of the chain changes more slowly than with smaller values of k, which means that it takes more iterations for the mean of the chain to become significantly different from the mean of the combined data. However, it also takes more iterations to get back to the center of the training data once the two means are no longer close.

Figures 2, 3, and 4, respectively, show examples of control boundaries from the *k*LINK (with $k = 1$) algorithm, T^2 statistics, and the ranking depth algorithm with simplicial depth for the bivariate gamma distribution with 200 simulated in-control training data observations. If the observation is inside the colored area, it is in control; otherwise the observation would be treated as out of control. We can see that a lower α results in a larger in-control boundary. It can be also observed that the *k*LINK chart produced more flexible control boundaries than the T^2 and ranking depth charts. This implies that the *k*LINK chart can effectively control Type I and Type II error rates in nonnormal situations.

[Figure 2 about here]

[Figure 3 about here]

[Figure 4 about here]

3. Simulation Study

This section describes the *k*LINK control chart via a simulation study, and then compares its performance against Hotelling's T^2 charts, and ranking depth (with simplicial depth) charts. We used an R package (www.r-project.org) to perform the simulation. In particular, we used the function depth( ) from the R package "depth"[21] to implement ranking depth charts.

3.1 Simulation scenarios

Two bivariate probability distributions were generated for the simulation study. We generated the data from the bivariate normal distribution with the mean vector μ_{in} and the covariance matrix S_{in} as follows:

$$\mu_{in} = \begin{bmatrix} 0 & 0 \end{bmatrix},\ S_{in} = \begin{bmatrix} 3.284 & 1.109 \\ 1.109 & 3.284 \end{bmatrix}.$$

Further, we generated the data from the bivariate gamma distribution in which both the shape and the scale parameters were specified as one. This particular set of shapes was devised to test the robustness of our *k*LINK chart to nonnormality. Figure 1 illustrates the results of applying the *k*LINK algorithm with $k = 1$ to the bivariate gamma dataset. The number to the right of each observation is the ranking for that observation. Note that observations in the center of the plot score lowest, and observations closer to the fringes score highest. In addition, observations in dense areas tend to have similar scores, which is a desirable property of this nonparametric method.

To evaluate performance of each method, we generated 500 in-control training observations and 200 testing observations in which the first 180 observations are in control and the last 20 observations are out of control. To generate the out-of-control data, three types of shifts (N1, N2, and N3) were considered in the multivariate normal case, and three types of shifts (G1, G2, and

G3) were considered in the multivariate gamma case. For univariate cases, the process shifts are generally expressed in terms of standard deviation. However, this may not be applicable in multivariate cases because shifts involve more than one process variable. In multivariate cases, shifts can be usually expressed in terms of the following noncentrality parameter λ, a function of the magnitude of the shift δ and the estimated covariance matrix S_{in}:

$$\lambda = \sqrt{\delta' S_{in}^{-1} \delta} \tag{4}$$

In the present study, we assume the covariance matrix has not changed and remains constant. The summary of the simulation scenarios for multivariate normal and gamma distributions is described as follows:

- N1 (small shift): λ=1,
- N2 (medium shift): λ=2,
- N3 (large shift): λ=3,
- G1 (small shift): λ=1,
- G2 (medium shift): λ=2,
- G3 (large shift): λ=3.

Figures 5 and 6, (which visualize the simulated datasets in different scenarios) show that the separation between in-control and out-of-control observations becomes clearer as the degree of shift increases.

[Figure 5 about here]

[Figure 6 about here]

[Figure 7 about here]

As mentioned earlier, in order to construct the *k*LINK chart, we need first to determine parameter *k*. In general, one can try various *k* and select the best *k* that produces the smallest error rate. Table 1 shows Type I and Type II error rates of the *k*LINK charts with different α from the G2 scenario. To find the optimal *k*, we generated 100 preliminary testing observations in which the first 50 observations are in control and the last 50 observations are out of control. The results shows that similar Type I and Type II error rates were obtained for different values of *k*, implying that *k* does not play a significant role in constructing *k*LINK charts. In this paper, we use k = 5 for further analyses.

[Table 1 about here]

3.2 Construction of kLINK charts

We demonstrate here the *k*LINK charts using the simulated data. Figure 1 shows the *k*LINK chart in the G2 scenarios. The monitoring statistics are the 1- φ or the plausibility that the testing measurements are out of control. The size of the combined data is 501, and the control limit (the horizontal solid line) is the desired α, set here at 0.1 and 0.2. The values are reported as out of control if the corresponding monitoring statistics (1- φ) exceed the control limit (shown in Figure 8 in the horizontal solid line).

[Figure 8 about here]

3.3. Effect of the control limits in the kLINK chart

The control limits used in the *k*LINK chart were established by a user-specified α. Figure 9 illustrates this under multivariate normal and multivariate gamma scenarios, where actual Type I and Type II error rates in the *k*LINK charts vary with the user-specified α shown on the *x*-axis. We ran 500 replications for each scenario to obtain average Type I and Type II error rates. As expected, increases in α result in higher Type I error rates and lower Type II error rates. The *k*LINK chart yielded low Type I and Type II error rates when α was between 0.05-0.2. All scenarios provided the same trend; with the larger shift yielding lower Type II error rates compared with the same Type I error rates. Because the maximum standard error for Type I and Type II errors in any of the simulated scenarios is quite small (less than 0.006), 500 replications is sufficient to draw a reliable conclusion.

[Figure 9 about here]

3.4. Performance comparison

Our *k*LINK charts were compared with Hotelling's T^2 charts, and ranking depth charts. Figures 10 and 11, respectively, show the comparative results in the normal and gamma scenarios. We ran 500 replications for each chart to determine the average Type I and Type II error rates. Given the same Type I error rates, lower Type II error rates are considered a better.

In cases using the normal distribution, all three control chart techniques produced comparable results. However, when the gamma distribution was used, the *k*LINK charts outperformed the other two methods. Interestingly, in cases using the gamma distribution, the ranking depth chart based on a nonparametric approach performed worse than the T^2 control chart. This unexpected result can be explained by the earlier figures (Figures 2, 3, and 4) that show that the control boundary of the ranking depth chart is less flexible than the *k*LINK algorithm. Furthermore, the control boundary of the ranking depth control chart is not even as effective as the T^2 control chart in the gamma distribution, an example of a skewed distribution. The maximum standard error of the Type I and Type II error rates from 500 replications is 0.006.

[Figure 10 about here]

[Figure 11 about here]

4. Exponentially Weighted kLINK

Because a traditional T^2 control chart monitors and evaluates a current process based on the most recent measurement, it may be insensitive to small process shifts. Multivariate exponentially weighted moving average (MEWMA) charts, which accumulate information from previous measurements, were devised to provide robustness to nonnormality and increased sensitivity to small shifts[17, 19, 26]. Here we propose an exponentially weighted moving average (EWMA) version of the *k*LINK chart (i.e., EWMA-*k*LINK). The monitoring statistic Z_i can be computed from the following equation:

$$Z_i = \lambda(1-\varphi_i) + (1-\lambda)Z_{i-1}, \tag{5}$$

where λ is the smoothing parameter with a range between 0 and 1, and Z_i is the EWMA-*k*LINK for measurement *i*. The starting value Z_0 can be obtained from the average 1-φ from in-control training data. The control limits of EWMA-*k*LINK charts are the desired α. The EWMA-*k*LINK chart signals an alarm when Z_i exceeds the control limit.

Because *k*LINK charts performed quite well with medium and large shifts, our focus here is the performance of the EWMA-*k*LINK chart in detecting small shifts in a nonnormal scenario. Small- and medium-shift scenarios from bivariate gamma distribution data (G1 and G2) were used to compare the performances of the MEWMA, the *k*LINK and EWMA-*k*LINK charts. We used average run length (ARL) as a measure of performance. Two different types of ARL can be defined based on the condition of the process. In-control ARL (ARL_0) is defined as the expected number of measurements needed for the chart to detect a shift in the in-control state; out-of-control ARL (ARL_1) is the number of measurements expected to be necessary for the chart to detect a shift in the out-of-control state.

In our simulation, ARL_0 and ARL_1 were computed based on 500 replications with λ at 0.25. Here the parameter λ was chosen arbitrarily because the main purpose was not to find the optimal parameter of the EWMA control chart. Primarily, we prefer a procedure that provides a lower ARL_1, given a similar value of ARL_0. Both the *k*LINK and EWMA-*k*LINK charts were constructed with $k = 5$. Figure 12 shows that in all scenarios and at any given $ARL_{0,}$ the EWMA-*k*LINK charts produced a lower ARL_1 than the *k*LINK and MEWMA charts.

[Figure 12 about here]

5. Discussions

5.1 Effect of the sizes of training data

The size of the training data can affect the performance of *k*LINK charts. We found that the degree to which the size of the training data affects the performance is determined by the underlying data distribution. We studied the performance of six different sizes of training data (i.e., 100, 200, 300, 400, 500, and 1000) under medium shift in bivariate normal (N2) and bivariate gamma (G2) scenarios. The resulting average values of Type I and Type II errors from 100 replications of different sizes of training data were shown in Figure 13. The maximum standard error in this experiment is 0.004. It can be observed from Figure 13 (a) that the line (n=100) appeared to have a higher Type II error rate for the smallest tested α value, indicating that when the training data size is small, say 100 or less, and the process data follow the normal distribution, *k*LINK charts may produce higher Typer II error rate. On the contrary, in the gamma distribution case, Figure 13 (b) shows that all lines do overlap. This indicates that the performance of *k*LINK chart is not significantly affected by the size of the training data. Overall, our *k*LINK chart produced the stable result once we get to a size of about 200. Further, we want to point out that the small set of training data is often not an issue any longer with the larger quantities of data that are stored in modern systems.

[Figure 13 about here]

5.2 Effect of high-dimensional data

Here we generated 200 in-control training observations and 200 testing observations from the 10-dimensional gamma distribution. In the testing data, the first 180 observations are in control and the last 20 observations are out of control. Figure 15 shows the performance of the *k*LINK (with k = 5), T^2, and ranking depth (with Tukey depth) charts in terms of Type I and Type II error rates from 500 replications. The maximum standard error in this simulation is 0.0052, small enough to draw a reliable conclusion. For the ranking depth chart, we used Tukey depth because simplicial depth in the R package is limited for only two-dimensional data. In addition, Masse and Plante (2009) indicated that Tukey depth provides only approximate depth values in high-dimensional data sets. The result indicated that our *k*LINK charts outperformed both T^2 and ranking depth charts. Note that the ranking depth chart could not produce Type I error rates less than 0.6638. This is due to the limitation of ranking depth charts in high-dimensional data sets.

[Figure 14 about here]

6. Conclusions

We have presented a new nonparametric multivariate control chart technique (the *k*LINK chart) and compared it against competing methods under normal and nonnornal scenarios. The results demonstrated that our *k*LINK chart outperformed the ranking depth chart and the Hotelling's T^2 chart in cases of nonnormal situations, and all three methods performed comparably in situations of normal distribution. To increase its capability to detect small process shifts, we also developed an EWMA version of the *k*LINK chart. The simulation study showed that the EWMA-*k*LINK chart performed better than both the *k*LINK or MEWMA charts in detecting small shifts in nonnormal cases.

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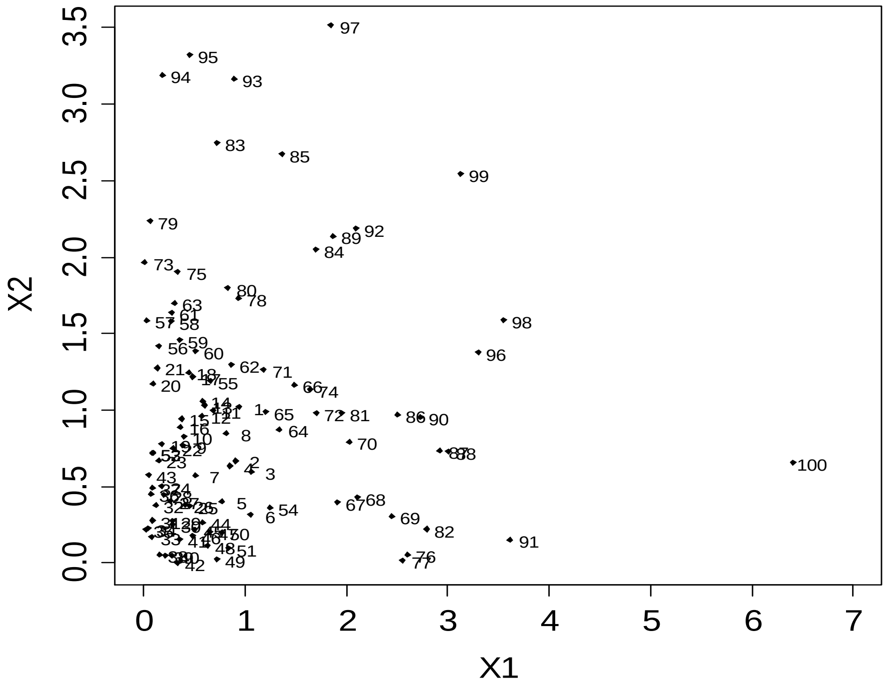
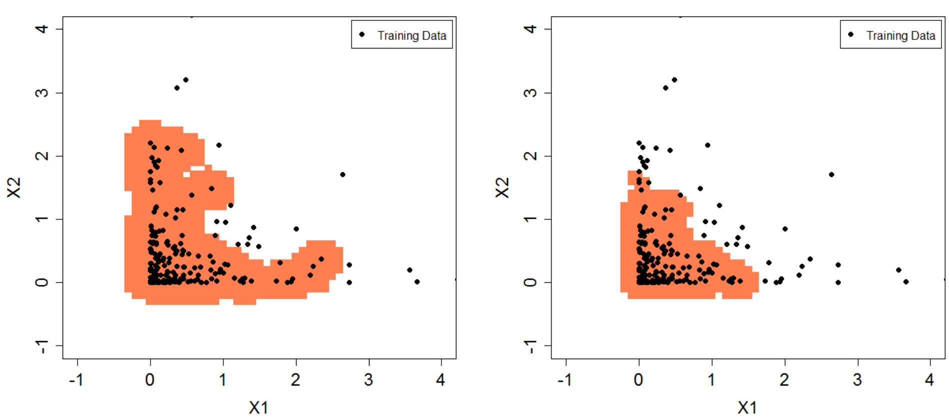
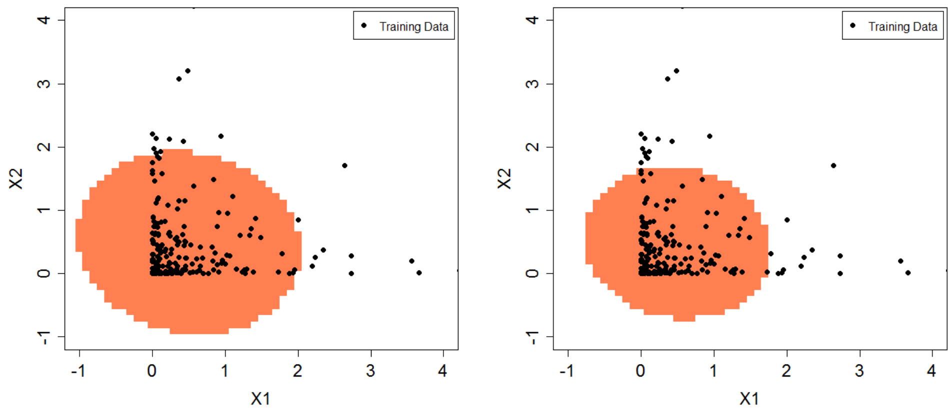


Figure 1. A *k*LINK algorithm with the parameter k =1, using the sample mean, applied to the bivariate gamma dataset. Numbers indicate the rankings.



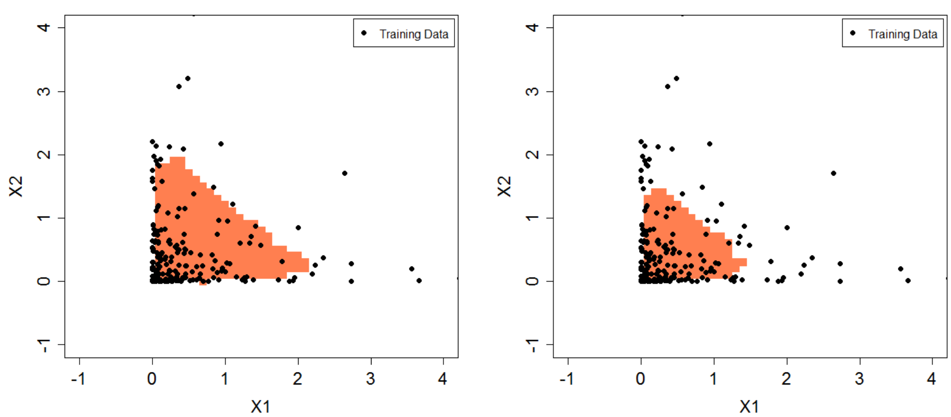
(a) (b)

Figure 2. Control boundaries of *k*LINK algorithm with k = 1 for bivariate gamma distribution with 200 in-control training data. (a) α = 0.1 and (b) α = 0.2



(a) (b)

Figure 3. Control boundaries of T^2 algorithm for bivariate gamma distribution with 200 in-control training data. (a) α = 0.1 and (b) α = 0.2.



(a) (b)

Figure 4. Control boundaries of the ranking depth algorithm for the bivariate gamma distribution with 200 in-control training data. (a) α = 0.1 and (b) α = 0.2.

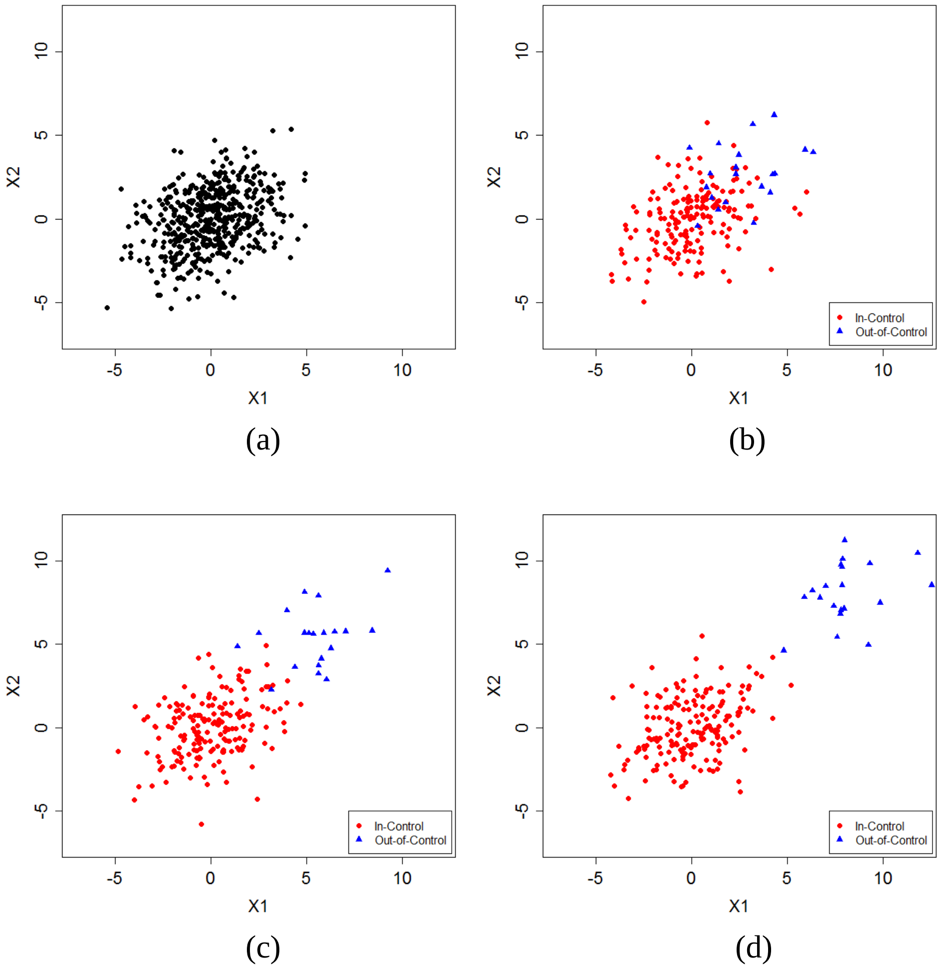


Figure 5. Simulation data from the bivariate normal distribution. (a) Training set, (b) N1, (c) N2, and (d) N3.

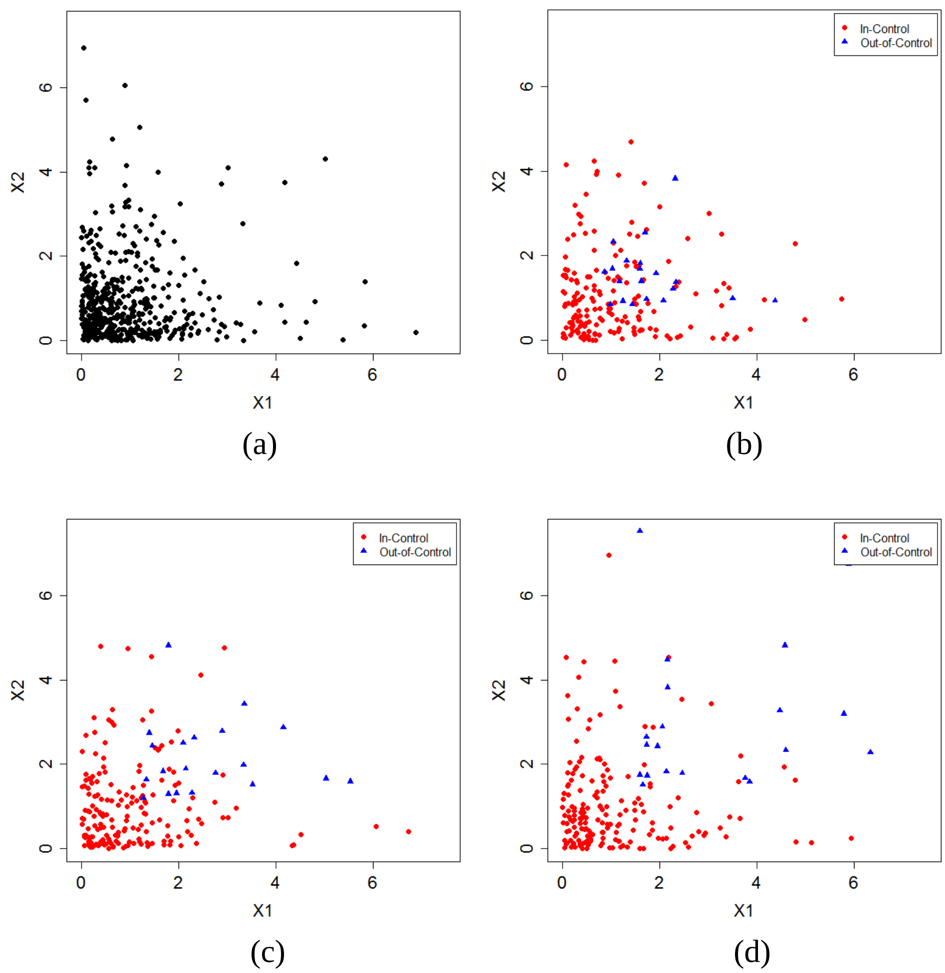


Figure 6. Simulation data from the bivariate gamma distribution. (a) Training set, (b) G1, (c) G2, and (d) G3.

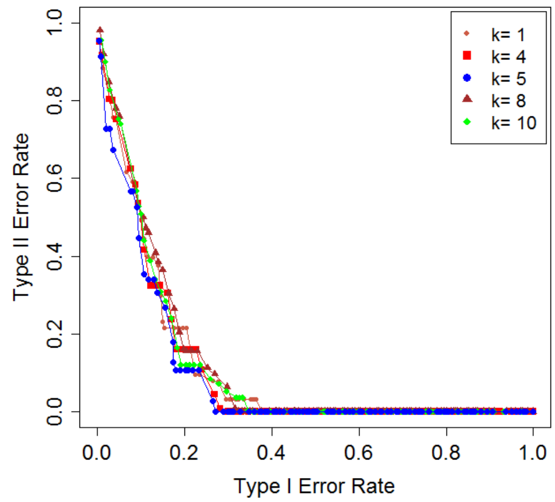


Figure 7. Determination of the appropriate *k* in the G2 scenario.

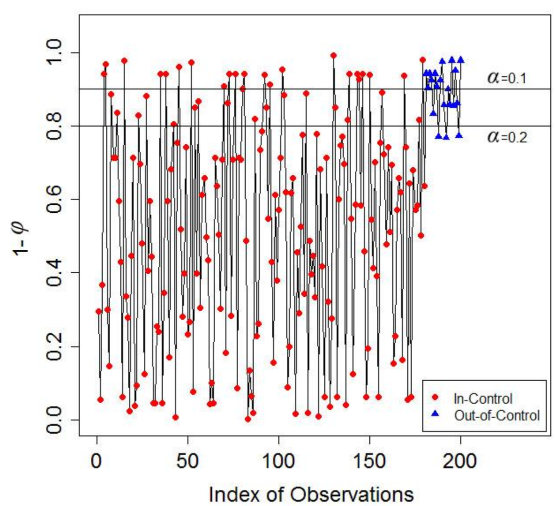
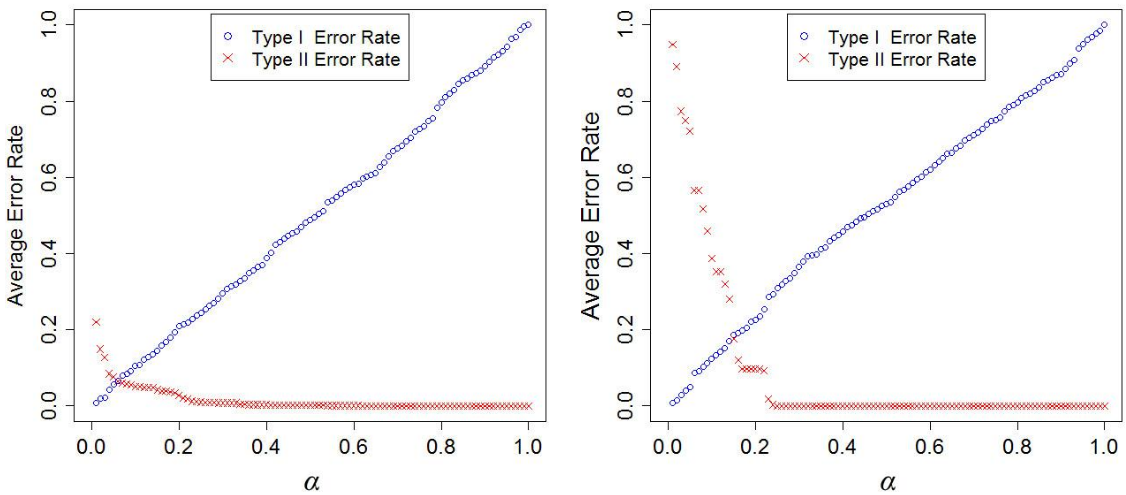
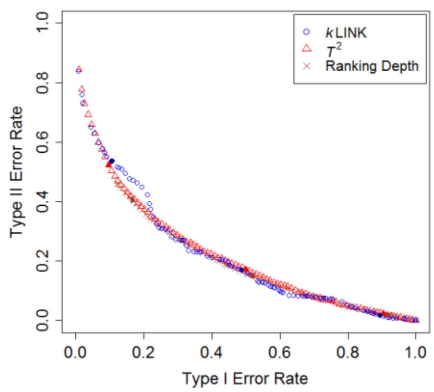


Figure 8. *k*LINK chart for the G2 scenario.

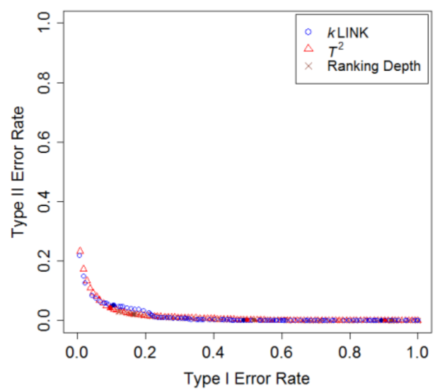


(a) (b)

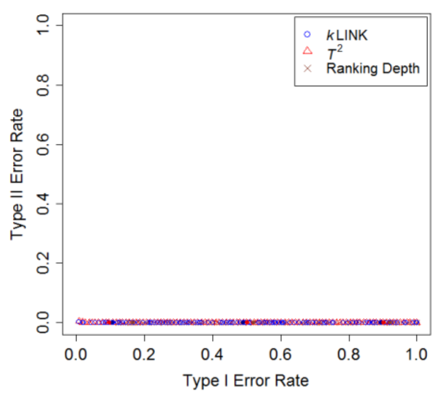
Figure 9. Average Type I and Type II error rates of the *k*LINK charts. (a) N2 scenario, (b) G2 scenario.



(a)

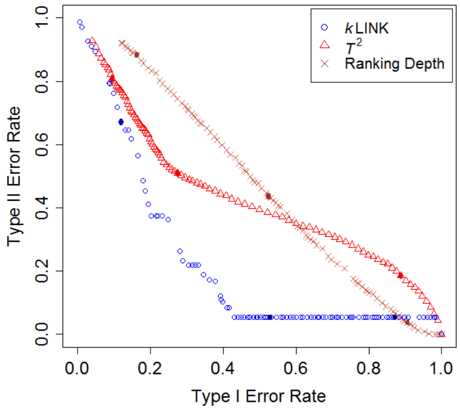


(b)

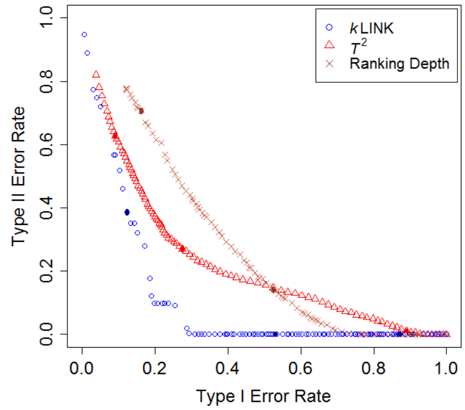


(c)

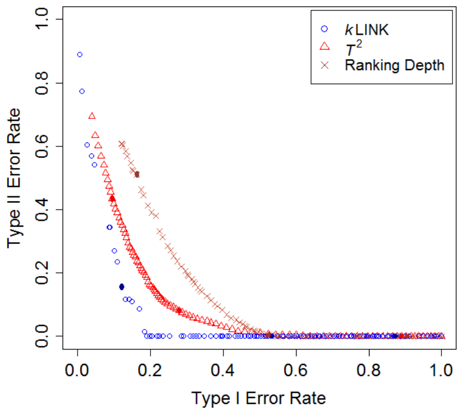
Figure 10. Actual Type I and Type II error rates of the *k*LINK, T^2, and ranking depth control



(a)

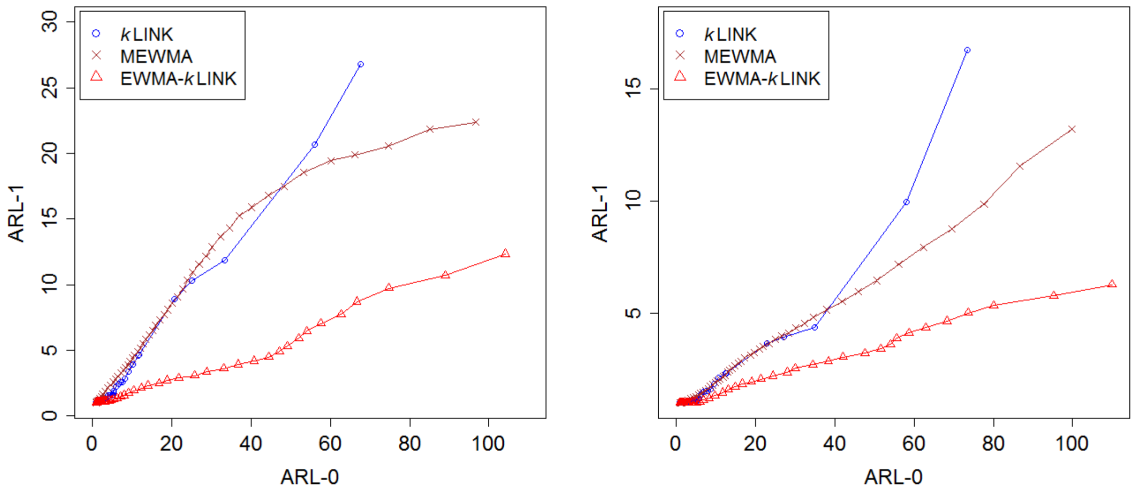


(b)



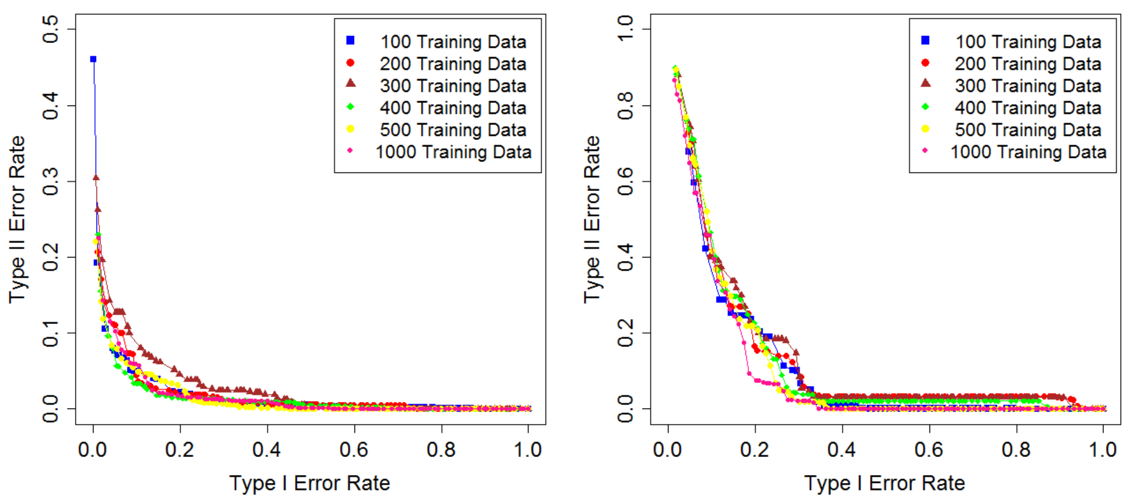
(c)

Figure 11. Actual Type I and Type II error rates of the *k*LINK, T^2, and ranking depth control



(a) (b)

Figure 12. Comparison of the average values of ARL_0 and ARL_1 among the EWMA-kLINK, *k*LINK, and EWMA-kLINK charts. (a) G1, and (b) G2.



(a) (b)

Figure 13. Average values of Type I and Type II error rates (from 100 replications) of the *k*LINK charts with 100, 200, 300, 400, 500, and 1000 training data for (a) bivariate normal (N2) and (b) bivariate gamma (G2) cases.

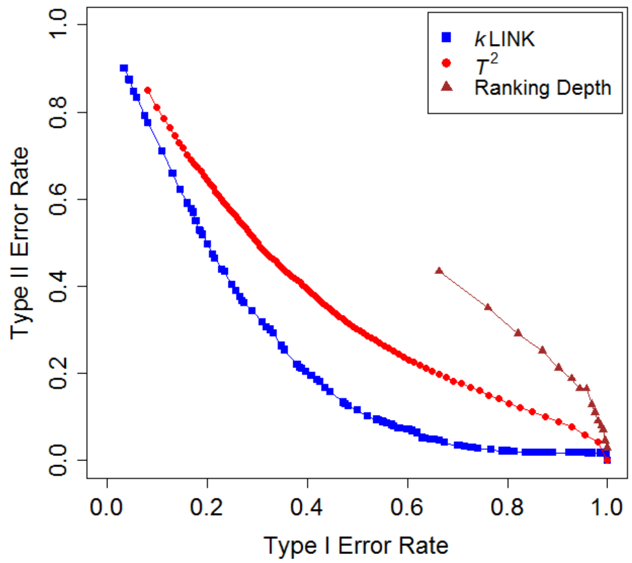


Figure 14. Actual Type I and Type II error rates of the *k*LINK (*k*=5), T^2, and ranking depth (with Tukey depth) charts from the 10-dimensional gamma distribution.

Table 1. Comparison of Type I and Type II error rates for different k and α in the G2 scenarios (Type II error rates are shown in parentheses)

	k										
α	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**S.D.**
0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(1.00)	(1.00)	(0.98)	(0.94)	(0.98)	(0.96)	(0.96)	(0.96)	(0.96)	(0.96)	(0.02)
0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	(0.94)	0.00
0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.88)	(0.84)	(0.88)	(0.84)	(0.84)	(0.84)	(0.84)	(0.84)	(0.84)	(0.84)	(0.02)
0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.72)	(0.78)	(0.82)	(0.82)	(0.80)	(0.80)	(0.80)	(0.80)	(0.80)	(0.80)	(0.03)
0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.70)	(0.72)	(0.78)	(0.76)	(0.76)	(0.76)	(0.78)	(0.78)	(0.78)	(0.80)	(0.03)
0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.60)	(0.58)	(0.60)	(0.58)	(0.58)	(0.56)	(0.52)	(0.50)	(0.50)	(0.54)	(0.04)
0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.58)	(0.58)	(0.60)	(0.58)	(0.58)	(0.56)	(0.50)	(0.50)	(0.50)	(0.50)	(0.04)
0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.58)	(0.58)	(0.56)	(0.56)	(0.52)	(0.48)	(0.48)	(0.48)	(0.48)	(0.48)	(0.05)
0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01
	(0.50)	(0.52)	(0.52)	(0.52)	(0.46)	(0.44)	(0.46)	(0.46)	(0.46)	(0.44)	(0.03)
0.10	0.02	0.00	0.02	0.00	0.02	0.02	0.02	0.02	0.02	0.02	0.01
	(0.44)	(0.42)	(0.42)	(0.40)	(0.36)	(0.36)	(0.36)	(0.36)	(0.36)	(0.40)	(0.03)
0.11	0.02	0.00	0.02	0.02	0.04	0.04	0.04	0.04	0.04	0.04	0.01
	(0.36)	(0.42)	(0.36)	(0.34)	(0.36)	(0.36)	(0.36)	(0.36)	(0.36)	(0.36)	(0.02)
0.12	0.02	0.02	0.04	0.02	0.04	0.04	0.04	0.04	0.04	0.04	0.01
	(0.36)	(0.34)	(0.34)	(0.34)	(0.36)	(0.36)	(0.36)	(0.34)	(0.34)	(0.36)	(0.01)
0.13	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.00
	(0.28)	(0.30)	(0.34)	(0.34)	(0.34)	(0.34)	(0.32)	(0.30)	(0.30)	(0.26)	0.00
0.14	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.00
	(0.24)	(0.30)	(0.28)	(0.26)	(0.26)	(0.26)	(0.24)	(0.24)	(0.24)	(0.24)	0.00
0.15	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.01
	(0.18)	(0.22)	(0.22)	(0.18)	(0.16)	(0.18)	(0.16)	(0.18)	(0.18)	(0.18)	(0.02)
0.16	0.06	0.06	0.04	0.06	0.06	0.06	0.04	0.04	0.04	0.04	0.01
	(0.16)	(0.22)	(0.16)	(0.12)	(0.14)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.04)
0.17	0.08	0.08	0.04	0.06	0.06	0.06	0.06	0.06	0.04	0.04	0.01
	(0.16)	(0.22)	(0.16)	(0.10)	(0.08)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.04)
0.18	0.08	0.10	0.06	0.06	0.06	0.06	0.06	0.06	0.04	0.04	0.02
	(0.16)	(0.22)	(0.16)	(0.10)	(0.08)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.04)
0.19	0.12	0.12	0.06	0.06	0.08	0.06	0.10	0.08	0.08	0.08	0.02
	(0.16)	(0.18)	(0.16)	(0.10)	(0.08)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.03)
0.20	0.14	0.12	0.10	0.10	0.08	0.10	0.12	0.12	0.10	0.12	0.02
	(0.16)	(0.18)	(0.16)	(0.10)	(0.08)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.03)