

A Multivariate Adaptive Regression Splines Cutting Plane Approach for Solving a Two-Stage Stochastic Programming Fleet Assignment Model. COSMOS Technical Report 10-06. The University of Texas at Arlington. Arlington, TX.

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Abstract

The fleet assignment model assigns a fleet of aircraft types to the scheduled flight legs in an airline timetable published six to twelve weeks prior to the departure of the aircraft. The objective is to maximize profit. While costs associated with assigning a particular fleet type to a leg are easy to estimate, the revenues are based upon demand, which is realized close to departure. The uncertainty in demand makes it challenging to assign the right type of aircraft to each flight leg based on forecasts taken six to twelve weeks prior to departure. Therefore, in this paper, a two-stage stochastic programming framework has been developed to model the uncertainty in demand, along with the Boeing concept of demand driven dispatch to reallocate aircraft closer to the departure of the aircraft. Traditionally, two-stage stochastic programming problems are solved using the L-shaped method. Due to the slow convergence of the L-shaped method, a novel multivariate adaptive regression splines cutting plane method has been developed. The results obtained from our approach are compared to that of the L-shaped method, and the value of demand-driven dispatch is estimated.

Keywords: Stochastic Programming, Airline Fleet Assignment model, L-shaped method

1. Introduction

In the airline industry, many airlines are confronted with increased competition from other carriers while they continue to address labor costs. Furthermore, high fuel costs have impacted the entire industry. Therefore, airlines try to find ways to reduce costs, increase profits, and improve load factors. One interesting option to reduce cost and increase revenues is to balance supply (seats) and demand (passengers). If an airline assigns an aircraft with too much capacity, flights will depart with empty seats. If an airline assigns an aircraft with insufficient capacity, this may cause lost (spilled) customers because of seat shortage. Consequently, airlines use a Fleet Assignment Model (FAM) in order to balance supply and demand. The objective of this model is to maximize profit (revenue minus operating costs). FAM has been credited for saving costs and improving airline operations. At American Airlines, improvement in operating margins increased 1.4% (Abara 1989). FAM has saved about \$15M of operating costs at US Airways (Rushmeier and Kontogiorgis 1997) and \$100M at Delta (Subramanian et al. 1994). The accuracy of cost and profit estimates is an important factor for the quality of a FAM solution. Cost estimates are relatively stable and known whereas revenue estimates depend on demand predictions. There are several

reasons to consider demand uncertain. It is challenging to assign the correct type of aircraft to each flight leg in the schedule that is published six to twelve weeks prior to the departure of the flight. One way of improving profit is to model demand stochastically and delay the fleet assignment decision closer to departure, and stochastic programming and the Boeing concept of demand driven dispatch concept of Boeing are very useful tools for this. Background knowledge for these two topics are described in Sections 1.1 and 1.2

1.1. Overview of Stochastic Programming

Stochastic Programming (SP) is a framework for modeling optimization problems that include uncertainty. SP involves uncertainty within decision making models and uses probability distributions for random events. The purpose of SP is to maximize the expectation function of the decisions based on random events. In this section, we present a two-stage SP with fixed recourse. In the first stage, the decisions are made without full information on some random event. Then, we solve the recourse problem (subproblem) based on the realization of random vectors and the decision from the first stage to get the second-stage decision.

The two-stage stochastic linear programming with fixed recourse can be formulated as :

$$\begin{aligned} \min \quad & z = \mathbf{c}^T \mathbf{x} + E_{\xi}[\min \mathbf{q}(\omega)^T \mathbf{y}(\omega)] \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} = \mathbf{b}, \\ & \mathbf{T}(\omega) \mathbf{x} + \mathbf{W} \mathbf{y}(\omega) = \mathbf{h}(\omega), \\ & \mathbf{x} \geq 0, \mathbf{y}(\omega) \geq 0, \end{aligned} \quad (1)$$

where $\mathbf{x} \in R^{n_1}$ is the first-stage decision vector, $\mathbf{y}(\omega) \in R^{n_2}$ is the vector of recourse or second-stage decision variables, $\mathbf{c} \in R^{n_1}$ is the known objective coefficient vector of \mathbf{x} , $\mathbf{q}(\omega) \in R^{n_2}$ is a coefficient matrix of vector $\mathbf{y}(\omega)$, and \mathbf{A} is an $m_1 \times n_1$ first-stage linear constraint matrix with the known right-hand side $\mathbf{b} \in R^{m_1}$. $\mathbf{T}(\omega)$ and \mathbf{W} are $m_2 \times n_1$ and $m_2 \times n_2$ matrices, respectively, specifying the second-stage linear constraints on \mathbf{x} and \mathbf{y} with the right-hand side vector $\mathbf{h}(\omega) \in R^{m_2}$. For a given realization of the stochastic variables, $\omega \in \Omega$, the second-stage problem data, $\mathbf{q}(\omega)$, $\mathbf{h}(\omega)$, and $\mathbf{T}(\omega)$, become known, and the second stage decision $\mathbf{y}(\omega, \mathbf{x})$ can be obtained. Let the vector $\xi^T(\omega)$ represent a scenario with the different components of the second stage, i.e., $\mathbf{q}(\omega)^T$, $\mathbf{h}(\omega)^T$, $\mathbf{T}(\omega)$, such that $\xi \in \Xi$, where Ξ represent a set of scenarios. The objective function in model (1) contains a deterministic term $\mathbf{c}^T \mathbf{x}$ and the expectation of the second-stage objective $\mathbf{q}(\omega)^T \mathbf{y}(\omega)$ taken over all realizations of the random event ω . We can formulate the second-stage value function for a given ω as

$$Q(\mathbf{x}, \xi(\omega)) = \min_y [\mathbf{q}(\omega)^T \mathbf{y} | \mathbf{W} \mathbf{y} = \mathbf{h}(\omega) - \mathbf{T}(\omega) \mathbf{x}, \mathbf{y} \geq 0]. \quad (2)$$

Let the expected second-stage recourse function be defined as: $\mathfrak{S}(\mathbf{x}) = E_{\xi} Q(\mathbf{x}, \xi(\omega))$, then the deterministic equivalent of the two-stage stochastic linear program can be written as

$$\begin{aligned} \min \quad & z = \mathbf{c}^T \mathbf{x} + \mathfrak{S}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq 0. \end{aligned} \quad (3)$$

One major challenge is determining the expected recourse function $\mathfrak{S}(\mathbf{x})$. Each evaluation of $\mathfrak{S}(\mathbf{x})$ at a given vector of \mathbf{x} requires solving many linear programming problems of the form

in (2). For complicated problems, such as fleet assignment, the iterative approximation methods described by Berge and Hopperstad (1993) can be very slow to converge because they normally require a large number of scenarios to sufficiently represent the stochasticity of the second-stage optimization problem in (2). Therefore, in each optimization iteration of the first stage, there is a high computational cost for evaluating $\mathfrak{S}(\mathbf{x})$ in the second stage, making the problem defined in model (3) even harder to solve. Chen (2001) proposed discretization of the \mathbf{x} -space to a finite set of points and solving for $\mathfrak{S}(\mathbf{x})$ only at those points, followed by a function approximation technique to estimate the entire surface of $\mathfrak{S}(\mathbf{x})$ in order to control the computational requirement. This approximation, $\hat{\mathfrak{S}}(\mathbf{x})$, will be computationally trivial to evaluate in model (3). For discretization, it is necessary to choose only those \mathbf{x} values that result in a feasible solution in the second stage.

1.2. Demand Driven Dispatch

For crew scheduling and maintenance planning, many airlines specify their schedules in advance. Since airline schedules must be set early enough to allow ample time to create crew and maintenance plans, these plans must be fixed at least six to twelve weeks prior to departure. A more robust FAM method would include reallocation of aircraft much closer to departure since most demand for flights is realized after the schedule is published. Berge and Hopperstad (1993) proposed that the concept of demand driven dispatch (D^3) be used for matching demand to aircraft close to the departure of the flight. This method use two stages of decision making. The first stage occurs six to twelve weeks prior to the departure of the flight, when the flight schedules are published. During this stage, *crew compatible families* of aircraft (see in Pilla et al. (2008)) are assigned to flights in the airline timeable. If they have the same cockpit model, the same crew group could operate either aircraft type. Therefore, two aircraft are said to be *crew compatible*. The second stage happens two weeks before the departure of flights when most of the demand is known and individual aircraft are assigned within the crew compatible families based on demand. In the second stage, swapping can occur with the assignment of specific flights in the second stage. For example, Boeing 757 and 767 models are crew compatible, and a Boeing 767 has more passenger seats than a Boeing 757. Suppose Flights A and B are firstly assigned to a 757 and 767, respectively. Demand is higher than expected for flight A, while flight B has realized a lower demand than expected two weeks before departure. In this case, the airline can swap the 757 and 767 without affecting the crew schedule. Because of this swapping, airlines can earn more revenue. Berge and Hopperstad (1993) stated that profits increase 1-5% by using this idea.

1.3. Literature Review

The airline fleet assignment problem has been a popular research topic for the several years and has been credited for increasing profits in many airlines. Sherali et al. (2006) described an extensive review of various fleet assignment ideas, models, and algorithms. Major airlines increased their flight schedules significantly and developed hub-and-spoke networks after the deregulation of the US airline industry in 1978. This allowed them to provide more destinations with a lot of traffic at the hubs making the fleet assignment problem much more complicated to solve. Farkas (1996) demonstrated that Revenue Management (RM) has a great impact on passenger volume and mix. Abara (1989) presented the first significant FAM application using an integer linear programming model. Moreover, he first solved the FAM LP relaxation, fixed variables, and then solved a Mixed Integer Programming (MIP) model. In order to determine solution quality, Abara (1989) noted that the number of high demand legs, which were covered by larger aircraft, increased from 76% to 90% and increased operating margin by 1.4%.

Hane et al. (1995) showed that a fleet assignment problem can be solved as a multi-commodity network flow problem, which formed the basis for a large portion of later FAM research. Moreover, they constructed the FAM problem as an MIP model and used a time line network for each airport and fleet type combination to formulate the constraints. Since these problems are often degenerate, they proposed different methods, which include cost perturbation, dual steepest edge simplex, an interior-point algorithm, model aggregation, branching on set-partitioning constraints, and prioritizing the order of branching to decrease the time required to solve the problem. Hane et al. (1995) reported run times twice as fast as a standard LP based branch-and-bound algorithm.

Subramanian et al. (1994) solved the fleet assignment problem at Delta Airlines by implementing a coldstart model with many of the features described by Hane et al. (1995). The optimization model did not require any initial fleet and produced results based on a raw schedule, providing the name *coldstart*. Talluri (1996) developed algorithms to solve a warmstart model that considers an existing daily fleet assignment and then tries to improve it by using local swap opportunities. Moreover, Gu et al. (1994) presented theoretical properties of the FAM problem. The solution of FAM affects subsequent planning decisions like aircraft routing, crew scheduling, and maintenance requirements. Consequently, extensions of FAM were considered in later research.

FAM was extended in Clarke et al. (1996) to address both maintenance and crew considerations. Rushmeier and Kontogiorgis (1997) presented a FAM formulation to manage aircraft routing issues. Barnhart et al. (1998) used the term *strings* to represent the assignment of a sequence of legs to a single aircraft. They showed a single model and solution method to simultaneously solve the fleet assignment and aircraft routing problems. Because of the possibility of numerous strings, Barnhart et al. (1998) proposed a delayed column generation technique to generate maintenance feasible routing solutions. Using a string based model, Rosenberger et al. (2003) developed a robust FAM that creates short hub-based strings, called *rotation cycles*, so that flight cancellations and delays are less likely to disrupt the entire network.

Modeling the objective function is a significant component of a successful FAM. The objective function of FAM can be maximizing revenue or minimizing passenger spill (lost revenue due to assigning smaller aircraft), or minimizing the number of aircraft being used. For the objective, the FAM models mentioned previously assume the demands for the different flight legs are independent and deterministic. In practice, both the assumptions are invalid. In a multi-leg itinerary, capacity of one flight leg affects the revenues of others, and the demand forecasts made early in the planning process can have significant errors. Hence, FAM can provide sub-optimal solutions by ignoring these effects.

Kniker (1998) augmented FAM with a Passenger Mix Model (PMM) to capture multi-leg passenger itineraries. Given a schedule with known flight capacities and a set of passenger demands with known fares, the combined FAM and PMM determines optimal demand and fleet assignments to maximize revenue. This is also referred to as Origin Destination (O-D) FAM. PMM assumes that demand is deterministic and that the airline has complete knowledge and control of which passengers they accept. Kniker (1998) solved the problem using branch-and-price and utilized sophisticated preprocessing techniques to reduce the computation involved. Barnhart et al. (2002) developed an alternate model to solve the O-D FAM problem addressed by Kniker (1998) using decision variables that assign a subset of legs to a fleet. They showed that by carefully selecting subsets, the model is computationally tractable.

Jacobs et al. (1999) presented an O-D FAM stochastic formulation and used the L-shaped method to solve it. Given an assignment solution, the revenue is estimated in the RM subproblem. The revenue function is approximated in the master problem with a series of L-shaped cuts with

each cut improving the accuracy. When a specified accuracy is reached in the relaxed master problem, the assignment variables are changed to integer and an MIP is solved. Although this method addresses both passenger flows in the network and demand uncertainty, Smith (2004) states that this approach can suffer from slow convergence and high fractionality.

Listes and Dekker (2002) used dynamic allocation for determining an optimal airline fleet composition. Given an airline schedule and a set of aircraft types, the fleet composition problem determines the number of aircraft of each type the fleet requires in order to maximize profit. They developed a two-stage SP model to determine a single fleet composition that maximizes profit in the first stage across all demand scenarios generated in the second stage. Similar to Berge and Hopperstad (1993), in the second stage, they solved a deterministic FAM model for each demand scenario allowing swapping to occur, and they employed a scenario aggregation approach, which was presented by Rockafellar and Wets (1991), to solve fleet assignment and decrease the computational complexity. They report a 90% runtime reduction and profit benefits up to 0.5 margin points. Even though the authors model the stochasticity of demand, the network effects are still ignored.

Recently, Sherali and Zhu (2008) developed a two-stage stochastic mixed-integer programming approach in which the first stage focuses on initial fleet assignment. Then, the second stage performs subsequent family-based type-level assignments according to forecasted market demand realizations like D^3 . They conducted polyhedral analysis of the proposed model and developed solution methods based on the L-shaped method.

In Pilla et al. (2008), the FAM problem was constructed by using a two-stage SP framework. Following a statistical perspective proposed by Chen (2001), Pilla et al. (2008) fitted a Multivariate Adaptive Regression Splines (MARS) (Friedman 1991) approximation for the expected profit function. However, they did not use the approximation function to optimize the two-stage SP FAM.

1.4. Contribution

The primary contribution of this paper is a MARS cutting plane algorithm (MARS-CP) that uses the approximation function from Pilla et al. (2008) to optimize the two-stage SP FAM. To do so, we revise the two-stage SP FAM formulation, and we derive a gradient of the MARS function, which is used to generate revenue cuts. We also compare MARS-CP with the traditional L-shaped method, similar to the one in Sherali and Zhu (2008). Finally, we estimate the value of using the D^3 concept.

The remainder of this paper is organized as follows. Section 2 presents the two-stage SP FAM. In Section 3, MARS-CP is discussed in detail. In Section 4, we describe the L-shaped method for the two-stage SP FAM problem. Section 5 presents computational results of implementations of MARS-CP and the L-shaped method. Finally, Section 6 concludes this paper with a discussion on future research.

2. Two-Stage SP FAM Formulation

In Section 2.1, we summarize the two-stage SP FAM from Pilla et al. (2008), and we revise it for MARS-CP and the L-shaped method in Section 2.2.

2.1. Original Two-Stage SP FAM

Given an airline schedule and a set of fleets of different aircraft that can fly each flight leg, the fleet assignment problem allocates the fleet of aircraft to the scheduled flights subject to the following operational constraints:

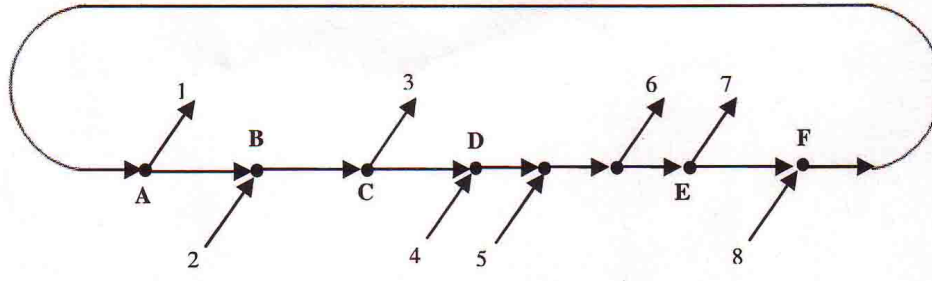


Figure 1: Time Line

- **Cover:** Each flight in the schedule must be assigned to exactly one aircraft type.
- **Balance:** Aircraft cannot appear or disappear from the network.
- **Plane Count:** The total number of aircraft assigned cannot exceed the number of available aircraft in the fleet.

The objective is to find a feasible assignment to maximize profit. Similar to models in Berge and Hopperstad (1993) and Hane et al. (1995), the fleet assignment problem is formulated as an integer multicommodity flow problem on a time line network. A time line is a graph that represents the arrival and departure events occurring at each station over a specified time period as shown in Figure 1.

Flights above the time line indicate departures, and flights below the time line indicate arrivals. In addition, the numbers indicate the corresponding flight legs. A *node* in a time line starts on an arrival and ends before the next arrival with at least one departure in between. In Figure 1, BC, DE and FA represent nodes. *Ground arc* is the arcs that connect within these nodes, *overnight arc* is the arc that connects the last arrival on the time line to the first departure. These arcs denote at least one plane being on the ground at a station and are defined as continuous variables. Because once all flight variables are integral, the values corresponding to these arcs will be integral as well. Any flight, which arrives at a particular station, will not be available for departure immediately because of the required time for fueling, cleaning, and loading passengers/baggage etc. As such, a *turn time* is added to all the arrivals before they are ready to take off. The turn time is dependent on the particular fleet type and the station. The sum of all the times corresponding to the arcs represent the *total ground time* of the planes at that particular station. The overnight arc includes a *plane count hour* (in general 4 A.M. EST) that is used for implementing the plane count constraints.

From Pilla et al. (2008), the assumptions for solving the two-stage SP FAM can be stated as follows:

- **Assumption 1:** The mean and variance for each itinerary-fare class are known.
- **Assumption 2:** Spilled passengers are assumed to be lost by the carrier and are not recaptured.
- **Assumption 3:** The D^3 swapping assignment variables are relaxed to be continuous.
- **Assumption 4:** Demand is known during D^3 swapping. Practically, demand may change even after swapping. Nevertheless, we assume that the airlines solve a deterministic D^3 swapping problem with expected demand because no more swapping opportunities are available.
- **Assumption 5:** Fleet types within the same family can always be swapped.

Let F denote the set of fleet types (indexed by f). Let L be the set of flight legs (indexed by l), and let H be the set of crew-compatible families (indexed by h), which can be used for each leg $l \in L$. Let e be a mapping from F to H ($e : F \rightarrow H$). Because we assign crew-compatible families in the first stage, for each leg $l \in L$ and for each crew-compatible family type $h \in H$, let a binary variable x_{hl} be defined such that

$$x_{hl} = \begin{cases} 1 & \text{if crew-compatible family } h \text{ is assigned to flight leg } l, \\ 0 & \text{otherwise.} \end{cases}$$

Let x_H represent the vector of first-stage binary variables. In the second stage, we assign specific aircraft within the crew-compatible family. As such, for each leg $l \in L$, for each aircraft $f \in F$, and for each scenario $\xi \in \Xi$, let a binary variable x_{fl}^ξ be defined such that

$$x_{fl}^\xi = \begin{cases} 1 & \text{if aircraft } f \text{ is assigned to the leg } l \text{ for scenario } \xi, \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to these variables as the D^3 swapping assignment variables, and we let x^ξ represent the vector of D^3 swapping assignment variables in scenario ξ . Because FAM and PMM are combined in this paper, let the decision variable z_i^ξ represent the number of booked passengers for itinerary-fare class i for scenario ξ . For additional notation of the two stage SP FAM, let

- S be the set of stations, indexed by s ,
- I be the set of itinerary-fare classes, indexed by i ,
- V be the set of nodes in the entire network, indexed by v ,
- $h(v)$ be the crew-compatible family associated with node v ,
- $f(v)$ be the fleet type associated with node v ,
- A_v be the set of flights arriving at node v ,
- D_v be the set of flights departing at node v ,
- U_f be the number of aircraft of type f ,
- U_h be the number of aircraft in crew-compatible family h ,
- f_i be the fare for itinerary-fare class i ,
- C_{fl} be the cost if aircraft type f is assigned to flight leg l ,
- a_{v+}^ξ be the value of the ground arc leaving node v in scenario ξ ,
- a_{v-}^ξ be the value of the ground arc entering node v in scenario ξ ,
- a_{v+} be the value of the ground arc leaving node v in the first-stage,
- a_{v-} be the value of the ground arc entering node v in the first-stage,
- O_f be the set of arcs that include the plane count hour for fleet type f , indexed by o ,
- O_h be the set of arcs that include the plane count hour for crew-compatible family h , indexed by o ,

- L_0 be the set of flight legs in the air at the plane count hour,
- Cap_f be the capacity of aircraft f ,
- D_i^ξ be the demand for itinerary-fare class i in scenario ξ .

The two-stage formulation can be written as:

$$\max \theta = E_\xi \left[- \sum_{l \in L} \sum_{f \in F} C_{fl} x_{fl}^\xi + \sum_{i \in I} f_i z_i^\xi \right] \quad (4)$$

$$\text{s.t. } \sum_{h \in H} x_{hl} = 1 \quad \forall l \in L, \quad (5)$$

$$\sum_{f \in h} x_{fl}^\xi = x_{hl} \quad \forall l \in L, h \in H, \xi \in \Xi, \quad (6)$$

$$a_{v^-}^\xi + \sum_{l \in A_v} x_{f(v)l}^\xi - \sum_{l \in D_v} x_{f(v)l}^\xi - a_{v^+}^\xi = 0 \quad \forall v \in V, \xi \in \Xi, \quad (7)$$

$$\sum_{o \in O_f} a_o^\xi + \sum_{l \in L_0} x_{fl}^\xi \leq U_f \quad \forall f \in F, \xi \in \Xi, \quad (8)$$

$$\sum_{i \ni l} z_i^\xi - \sum_{f \in F} Cap_f x_{fl}^\xi \leq 0 \quad \forall l \in L, \xi \in \Xi, \quad (9)$$

$$0 \leq z_i^\xi \leq D_i^\xi \quad \forall i \in I, \xi \in \Xi, \quad (10)$$

$$x_{fl}^\xi \in \{0, 1\} \quad \forall f \in F, l \in L, \xi \in \Xi, \quad (11)$$

$$x_{hl} \in \{0, 1\} \quad \forall l \in L, h \in H, \quad (12)$$

$$a_{v^+}^\xi \geq 0 \quad \forall v \in V, \xi \in \Xi. \quad (13)$$

The objective (4) is to maximize profit by assigning aircraft within the crew-compatible allocation made in the first stage. In two-stage SP FAM, cover constraints are required for both stages. Constraint set (5) includes the *first-stage cover constraints* that guarantee each flight is assigned to a crew-compatible family. The constraints in set (6), which represent the *second-stage cover constraints*, ensure that flight legs are reassigned within the crew-compatible family originally assigned within the first stage. The *balance constraints* (7) are needed to maintain the circulation of aircraft throughout the network. Then, we need to count the number of aircraft of each fleet being used to formulate the *plane count constraints* (8). As such the ground arcs that cross the plane count hour and the flights in air during that time are summed to assure that the total number of aircraft of a particular fleet type does not exceed the number available. Constraints (9) limit the number of booked passengers on different itineraries for a flight l to the capacity of the aircraft assigned, and constraint set (10) limits the number of passengers over a fare class to the forecasted demand. Constraint sets (11) and (12) require that the assignment variables be binary. As noted in Assumption 3, the D^3 swapping assignment variables in (11) will be relaxed to be continuous; that is, constraint set (11) are replaced by

$$x_{fl}^\xi \geq 0, \quad \forall f \in F, l \in L, \xi \in \Xi. \quad (14)$$

In practice most crew-compatible families include only one or two aircraft types, so integer solutions result (Berge and Hopperstad 1993). An upper bound on the objective function can be obtained for families with more than two aircraft types.

2.2. Revised Two-Stage SP FAM

In the original two-stage SP FAM problem, the objective function and the constraints form a block angular structure, and the standard L-shaped method can be applied to solve the problem. The master problem makes a crew-compatible assignment, and each recourse subproblem solves a particular D^3 swapping assignment within the family assigned in the first stage. The revised two-stage SP FAM is represented as follows:

$$\begin{aligned}
 & \max \eta & (15) \\
 \text{s.t. } & a_{v^-} + \sum_{l \in A_v} x_{h(v)l} - \sum_{l \in D_v} x_{h(v)l} - a_{v^+} = 0 & \forall v \in V, & (16) \\
 & \sum_{o \in O_h} a_o + \sum_{l \in L_0} x_{hl} \leq U_h & \forall h \in H, & (17) \\
 & a_{v^+} \geq 0 & \forall v \in V, & (18) \\
 & \eta \leq \theta(x_H), & (19) \\
 & x_H \text{ satisfies (5) and (12),}
 \end{aligned}$$

where

$$\begin{aligned}
 \theta(x_H) = \max E_\xi & \left[- \sum_{l \in L} \sum_{f \in F} C_{fl}(x_{fl}^\xi) + \sum_{i \in I} f_i z_i^\xi \right] & (20) \\
 \text{s.t. } & (x^\xi, a^\xi) \text{ satisfies (6), (7), (8), (9), (10), (13), and (14).}
 \end{aligned}$$

The master problem is (5), (12), and (15) - (19), while (6), (7), (8), (9), (10), (13), (14), and (20) represent the D^3 swapping recourse subproblem. Most of the revised two-stage SP FAM is a straightforward application of the deterministic equivalent of two-stage stochastic programming described in Section 1.1. However, constraints (16), (17), and (18) are the first-stage constraints, which were not explicitly present in the original two-stage SP FAM formulation. Nonetheless, they can be shown to be valid by aggregating the second-stage constraints within the crew-compatible families. We will refer to constraints (16) and (17) as the *first-stage balance constraints* and the *first-stage plane count constraints*, respectively. With Assumption 5, we know that for any x_H that satisfies the first-stage constraints, there exists a feasible assignment to the D^3 swapping recourse subproblem. By using these constraints, we can avoid infeasibility issues with the second-stage recourse function $\theta(x_h)$. Constraint (19) ensures that the objective value is no more than the function $\theta(x_h)$. The D^3 swapping recourse subproblem can be decomposed by each scenario and solved as in the standard L-shaped method. In both our MARS-CP approach and in the L-shaped method, constraint (19) is replaced with linear cutting planes.

3. MARS-CP

In this section, we describe MARS-CP. Section 3.1 overviews our statistical approach based on Design and Analysis of Computer Experiments, and we describe how the MARS function was fit in Pilla et al. (2008) in Section 3.2. In Section 3.3, we derive the gradient of the MARS function, and Section 3.4 explains how the MARS function can be optimized using a cutting plane algorithm.

3.1. Design and Analysis of Computer Experiments

Design and Analysis of Computer Experiments (DACE) can be used to optimize a complex system (Chen et al. 2006; Tsai and Chen 2005). In a DACE approach, a computer model is used to study a complex system. Design of Experiments (DOE) is used to specify inputs for a set of computer model runs for which performance outputs are observed. Using these data, a meta-model approximates the relationship between *input variables* (vector x) and an *output variable* y . In the DACE literature, the computer model is typically a simulation model. However, DACE methods have been developed for solving stochastic dynamic programming (Chen et al. 1999) and Markov decision problems (Chen et al. 2003) effectively, and Chen (2001) first suggested applying DACE to stochastic programming.

Figure 2 diagrams our DACE-based approach, where the computer model consists of the linear programs that yield the recourse function θ in equation (20) of SP-FAM. The MARS-CP algorithm developed in this paper is a DACE approach to solve a two stage SP-FAM. The approach is depicted in Figure 2. The left box in the figure is the DACE Phase, which was the subject of Pilla et al. (2008), while the right box uses the MARS-CP algorithm in the Optimization Phase. In the DACE Phase, since the first stage requires only the crew-compatible allocation (CCA) of aircraft, a reduced state space corresponding to the crew group allocation is generated. Within this state space ($x = \text{CCA}$), DoE can be used to select certain discretization points. Scenarios are generated based on known probability distributions of passenger demand, and for each CCA, the objective values of the LP relaxations of the second-stage FAM are collected and averaged as the computer model output variable θ . A MARS statistical model, $\hat{\theta}(x)$ is fit to these data to generate an approximate second-stage recourse function, which can then be employed for more efficient future evaluations in the Optimization Phase and optimized by dynamically generating revenue cuts. The MARS-CP algorithm uses this MARS model $\hat{\theta}(x)$ to conduct quick evaluations of the recourse function in a cutting plane algorithm.

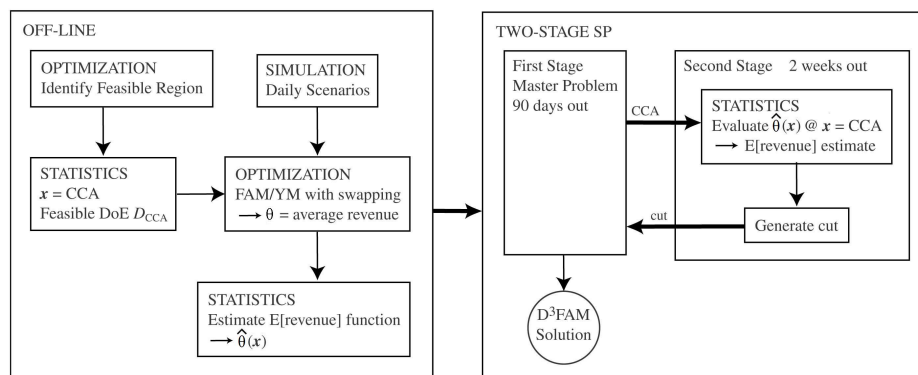


Figure 2: Two-Stage Stochastic Programming framework

Since a fit for the recourse function can be obtained well in advance of the six-to-ten week period, we have utilized two phases to further reduce the computation involved, as shown in Figure 2.

3.2. Design of Experiments and MARS

In this section, we summarize the approach to fit the MARS approximation to the recourse function in Pilla et al. (2008). The primary objective of the first-stage problem is to assign an initial

crew-compatible allocation (CCA). Consider the following *first-stage nonnegativity constraints* given by

$$x_{hl} \geq 0, \quad \forall l \in L, h \in H. \quad (21)$$

The experimental region in which the design is generated is the polytope defined by the continuous relaxation of the first-stage constraints P ; that is,

$$P = \{x_H | x_H \text{ satisfies (5), (16), (17), (18), and (21)}\}. \quad (22)$$

For conducting the DoE method, these constraints must be preprocessed such that they form a *polytope* defined by a system of linear inequalities $Ax \leq b$. Preprocessing to reduce the state space is employed by using the explicit equality constraints, as well as implicit equalities present in the constraints. Savelsbergh (1994) presented preprocessing techniques that can be used to reduce MIP problems. Let $P := \{x \in R | Ax \leq b\}$ be a nonempty convex *polytope* formed by the first-stage constraints. Discretization points within this *polytope* can be generated to represent the initial CCA, and any infeasible points created during the design can be projected onto the feasible polytope.

The MARS (Multivariate Adaptive Regression Splines) algorithm was developed by Friedman (1991) as a statistical modeling method for estimating a completely unknown relationship between a single output variable (typically observed with uncertainty) and several input variables. The MARS model is composed of a linear combination of basis functions; one univariate form can be seen in Figure 3. The truncated linear form is characterized by a "sign" ϕ and a single "knot" K at which the function bends. In general, the j th MARS basis function in the model is a product of T_j truncated linear functions:

$$B_j(x_H) = \prod_{l=1}^{T_j} [\phi_{l,j} \cdot (x_{v(l,j)} - K_{l,j})]_+, \quad (23)$$

where $x_{v(t,j)}$ is the predictor variable corresponding to the t th truncated linear function in the j th basis function, $K_{t,j}$ are knot locations at which the basis function bends, and $\phi_{t,j}$ is $+1$ or -1 . The MARS model is of the form

$$\hat{\theta}(x_H) = \beta_0 + \sum_{j=1}^M \beta_j \prod_{t=1}^{T_j} [\phi_{t,j} \cdot (x_{v(t,j)} - K_{t,j})]_+, \quad (24)$$

where β_0 is the coefficient for the constant basis function, M is the number of linearly independent basis functions, β_j is the coefficient of j^{th} basis function $\prod_{t=1}^{T_j} ([\phi_{t,j}(x_{v(t,j)} - K_{t,j})]) = B_j(x)$, and $x_{v(t,j)}$ is the predictor variable. Because the MARS model is a statistical linear model, the standard least squares formulas are applicable for estimating the model parameters β_j . The quintic form, also seen in Figure 3, is characterized by three knots and will be specified in the next section, where we derive the gradient of a MARS model.

The MARS algorithm is adaptive because the basis functions are selected based on the data. The original MARS algorithm of Friedman (1991) specifies a three-phase process. In the first phase, a model is grown by sequentially adding truncated linear basis functions that best improve the fit to the data until a "sufficient" number of terms have been added (user-specified by a parameter M_{max}). In the second phase, basis functions are deleted until a balance of bias and variance is found. The greedy search procedures simultaneously select input variables and knots. The third

phase converts the truncated linear basis functions to a smoother form. Friedman (1991) suggested a cubic form that enables a continuous first derivative, and Chen et al. (1999) recommended the quintic form we use in our research to additionally enable a continuous second derivative.

The run time for MARS is dependent upon the number of basis functions that the user specifies with the parameter M_{max} . Motivated by the fact that M_{max} requires some trial and error to identify, and that the second phase of the MARS algorithm is computationally slow, a computationally faster variant of MARS that automatically selects M and eliminates the second phase was developed by Tsai and Chen (2005). This MARS variant is what we employed in this paper.

3.3. MARS Gradient

To enable a continuous first and second derivative, Chen et al. (1999) replaced the truncated linear functions with quintic functions as shown in Figure 3. For sign ϕ and knots K_-, K, K_+ , the

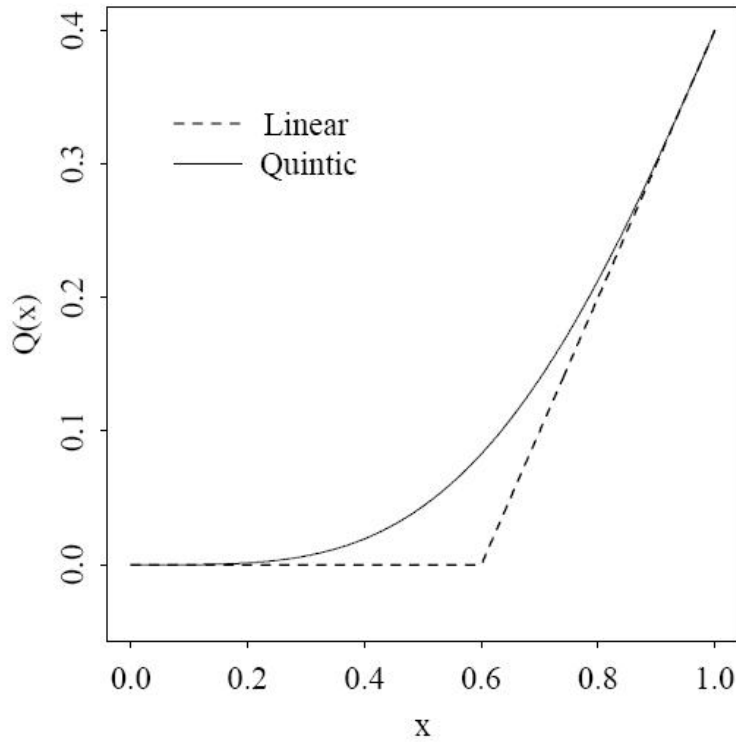


Figure 3: Continuous derivative MARS function

quintic functions are defined as:

$$Q(x|\phi = 1, K_-, K, K_+) = \begin{cases} 0, & x \leq K_- \\ \vartheta_+(x - K_-)^3 + \iota_+(x - K_-)^4 + \kappa_+(x - K_-)^5, & K_- < x < K_+ \\ x - K, & x \geq K_+, \end{cases} \quad (25)$$

and

$$Q(x|\phi = -1, K_-, K, K_+) = \begin{cases} K - x, & x \leq K_- \\ \vartheta_-(x - K_+)^3 + \iota_-(x - K_+)^4 + \kappa_-(x - K_+)^5, & K_- < x < K_+ \\ 0, & x \geq K_+. \end{cases} \quad (26)$$

where,

$$\vartheta_+ = \frac{6K_+ - 10K + 4K_-}{(K_+ - K_-)^3}, \quad (27)$$

$$\iota_+ = \frac{-8K_+ + 15K - 7K_-}{(K_+ - K_-)^4}, \quad (28)$$

$$\kappa_+ = \frac{3K_+ - 6K + 3K_-}{(K_+ - K_-)^5}, \quad (29)$$

$$\vartheta_- = \frac{(-1)(6K_- - 10K + 4K_+)}{(K_- - K_+)^3}, \quad (30)$$

$$\iota_- = \frac{(-1)(-8K_- + 15K - 7K_+)}{(K_- - K_+)^4}, \quad (31)$$

$$\kappa_- = \frac{(-1)(3K_- - 6K + 3K_+)}{(K_- - K_+)^5}. \quad (32)$$

Consider a univariate quintic MARS approximation represented as below.

$$\hat{\theta}(x) = \beta_0 + \sum_{j=1}^M \beta_j Q_j(x). \quad (33)$$

The first derivative is given by

$$d(\hat{\theta}(x))/dx = \sum_{j=1}^M \beta_j Q'_j(x), \quad (34)$$

where,

$$Q'_j(x|\phi = 1, K_-, K, K_+) = \begin{cases} 0, & x \leq K_- \\ 3\vartheta_+(x - K_-)^2 + 4\iota_+(x - K_-)^3 + 5\kappa_+(x - K_-)^4, & K_- < x < K_+ \\ 1, & x \geq K_+, \end{cases} \quad (35)$$

or

$$Q'_j(x|\phi = -1, K_-, K, K_+) = \begin{cases} -1, & x \leq K_- \\ 3\vartheta_-(x - K_+)^2 + 4\iota_-(x - K_+)^3 + 5\kappa_-(x - K_+)^4, & K_- < x < K_+ \\ 0, & x \geq K_+. \end{cases} \quad (36)$$

For a two factor interaction,

$$\hat{\theta}(x) = \beta_0 + \sum_{j=1}^M \beta_j Q_{j1}(x_1) Q_{j2}(x_2), \quad (37)$$

The gradient is given by the following:

$$\nabla \hat{\theta}(x) = \begin{bmatrix} \sum_{j=1}^M \beta_j Q'_{j1}(x_1) Q_{j2}(x_2) \\ \sum_{j=1}^M \beta_j Q_{j1}(x_1) Q'_{j2}(x_2) \end{bmatrix}. \quad (38)$$

3.4. Optimization of MARS Function

For the two-stage SP FAM, the constraints are all linear, and under the assumption that the MARS approximation $\hat{\theta}(x)$ is concave, a variant of Kelley's cutting plane method can be used to solve the problem.

Specifically, a set of linear constraints may be used to represent constraint (19) as follows:

$$\eta \leq \hat{\theta}(\bar{x}_H) + (x_H - \bar{x}_H)^T \nabla \hat{\theta}(\bar{x}_H), \quad \forall x \in P, \quad (39)$$

Constraint set (39) is similar to the optimality cuts in the L-shaped method, which are added dynamically in each iteration. Likewise, the cutting plane algorithm will add constraints in (39) dynamically, and we let \bar{P} represent a subset of points in P . Algorithm 1 presents the cutting plane algorithms to optimize the two-stage FAM problem using a MARS recourse function.

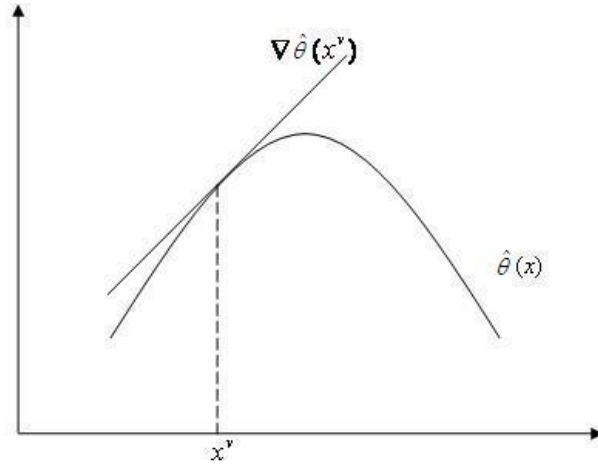


Figure 4: Cutting Plane Method using MARS approximation

Algorithm 1 Cutting Plane Algorithm for MARS

Step 0: Set $v \leftarrow 0$, $\eta^v \leftarrow \infty$ and let x_H^v be an initial feasible first-stage assignment. Go to Step 2.

Step 1: Set $v \leftarrow v + 1$. Solve the restricted master problem (RMP) represented by equations (5), (12), (15), (16), (17), (18), and (39) over the subset \bar{P} to get a solution (x_H^v, η^v) .

Step 2: Evaluate the MARS approximation $\hat{\theta}(x_H^v)$.

if $\eta^v \leq \hat{\theta}(x_H^v) + \epsilon$ **then**

 Stop. x_H^v is an optimal first-stage solution.

else

 Generate an optimality cut and set $\bar{P} \leftarrow \bar{P} \cup \{x_H^v\}$. Go to Step 1.

end if

4. L-shaped method for Two-Stage FAM Problem

In the two-stage SP FAM in Section 2.2, the objective function and the constraints form a block angular structure, and the traditional L-shaped method can be applied to solve the problem

for comparison to our MARS-CP approach. Sherali and Zhu (2008) have applied an L-shaped method algorithm for solving the two-stage SP FAM. While this section was written independently of Sherali and Zhu (2008), the formulation in this paper is similar to theirs. The D^3 swapping subproblem can be decomposed by each scenario, and finally an average over the scenarios is calculated to get the recourse function value θ . For a random realization ξ , let π_{hl}^ξ , γ_v^ξ , ρ_f^ξ , δ_l^ξ , and μ_i^ξ be the dual variables corresponding to the constraints (6), (7), (8), (9), (10), (11), and (13), respectively. For a given ξ and \bar{x}_H , the dual of the subproblem is:

$$\min \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi \bar{x}_{hl} + \sum_{f \in F} \sum_{\xi \in \Xi} \rho_f^\xi U_f + \sum_{i \in I} \sum_{\xi \in \Xi} \mu_i^\xi D_i^\xi \quad (40)$$

$$\text{s.t. } (\pi, \gamma, \rho, \delta, \mu) \in \Delta, \quad (41)$$

where Δ represents the polyhedron formed by the dual constraints.

The following inequality has to be satisfied at optimality for each $\bar{x}_H \in P$ and each $(\pi, \gamma, \rho, \delta) \in \Delta$ as defined by constraint (19):

$$\eta \leq \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi \bar{x}_{hl} + \sum_{f \in F} \sum_{\xi \in \Xi} \rho_f^\xi U_f + \sum_{i \in I} \sum_{\xi \in \Xi} \mu_i^\xi D_i^\xi; \quad (42)$$

otherwise, the master problem must be re-solved with an added L-shaped optimality cut. The optimality cut is represented as:

$$\eta \leq \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi x_{hl} + \sum_{f \in F} \sum_{\xi \in \Xi} \rho_f^\xi U_f + \sum_{i \in I} \sum_{\xi \in \Xi} \mu_i^\xi D_i^\xi. \quad (43)$$

The last two terms on the right hand side are constant, and they can be determined for any particular ξ , once the dual values π_{hl}^ξ , ρ_f^ξ and μ_i^ξ are known. For a given crew-compatible family solution \bar{x}_H , let $\bar{\theta}(\bar{x}_H)$ be the average recourse function value over all scenarios (ξ), then the last two terms can be calculated as:

$$\sum_{f \in F} \sum_{\xi \in \Xi} \rho_f^\xi U_f + \sum_{i \in I} \sum_{\xi \in \Xi} \mu_i^\xi D_i^\xi = \bar{\theta}(\bar{x}_H) - \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi \bar{x}_{hl}. \quad (44)$$

Now the L-shaped optimality cut represented by equation (43) can be modified as:

$$\eta - \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi x_{hl} \leq \bar{\theta}(\bar{x}_H) - \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi \bar{x}_{hl}. \quad (45)$$

Given this, the master problem can be reformulated as:

$$\max \eta \quad (46)$$

$$\text{s.t. } \eta - \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi x_{hl} \leq \bar{\theta}(\bar{x}_H) - \sum_{l \in L} \sum_{h \in H} \sum_{\xi \in \Xi} \pi_{hl}^\xi \bar{x}_{hl} \quad \forall \bar{x}_H \in P \quad (47)$$

$$x_H \text{ satisfies (5), (16), (17), and (18)}$$

As in MARS-CP, the first-stage constraints have been added to the master problem. As such, the complete recourse of the problem is maintained, and no feasibility cuts are needed. If not all of the first-stage constraints are added to the master problem, then the first-stage decision obtained might

not be second-stage feasible. In that case, feasibility cuts can be added to the master problem. In general, for a random realization ξ in which $(\pi, \gamma, \rho, \delta, \mu)$ is a dual extreme ray, the feasibility cut for the two-stage SP FAM is represented as:

$$\sum_{l \in L} \sum_{h \in H} \pi_{hl}^{\xi} x_{hl} + \sum_{f \in F} \rho_f^{\xi} U_f + \sum_{i \in I} \mu_i^{\xi} D_i^{\xi} \geq 0. \quad (48)$$

The L-shaped method for the two-stage SP FAM with complete recourse is presented in Algorithm 2.

Algorithm 2 L-shaped method algorithm for two-stage SP FAM

Step 0: Set $v \leftarrow 0$, $\eta^v \leftarrow \infty$ and let x^v be an intital feasible first-stage assignment. Let $\xi \in \Xi$ be the set of scenarios indexed by k . Go to Step 2.

Step 1: Set $v \leftarrow v + 1$. Solve the RMP represented by equations (5), (12), (15), (16), (17), (18), and (47) over the subset \bar{P} to get a solution (x_H^v, η^v) .

Step 2:

for all $k = 1, \dots, K$ **do**

Solve the subproblem D^3 swapping subproblem to obtain the recourse function value θ_k^v and the simplex multiplier $\pi_{fl}^{\xi_k}$.

end for

Calculate:

$$\theta^v(x_H^v) = \sum_{k \in K} p_k \theta_k^v \quad (49)$$

$$\pi_{hl}^{\xi} = \sum_{k \in K} \sum_{f \in h} p_k \pi_{fl}^{\xi_k} \quad (50)$$

where p_k is the probability associated with the k^{th} scenario.

if $\eta^v \leq \theta^v(x_H^v)$ **then**

Stop. x_H^v is the optimal first-stage solution.

else

Generate an optimality cut and set $\bar{P} \leftarrow \bar{P} \cup \{x_H^v\}$. Go to Step 1.

end if

4.1. Scenario Generation

Discrete distributions must be used to represent the scenarios in the two-stage SP FAM problems. The stochasticity of the demand is incorporated by generating demand scenarios for each itinerary-fare class. There are two major issues:

- The number of scenarios must be small enough for the two-stage SP FAM to be solvable.
- The number of scenarios must be large enough to represent the underlying distribution or data adequately.

5. Case Study Results

Our case study from a real airline carrier is the same as that described in Smith (2004) and Pilla et al. (2008), with a weekly schedule containing 50 stations, 2358 legs, seven fleet types and four crew compatible families. The three primary objectives for this research were to quantify:

- The importance of using demand driven dispatch.
- The value of a two-stage model with respect to a single-stage model.
- How the MARS-CP approach developed in this paper compares to L-shaped method.

In this section, our DACE and MARS-CP method described in Section 3 is compared with the L-shaped method described in Section 4.

A two-stage SP FAM problem was constructed similarly to the method discussed in Section 2.2 and solved by relaxing the second-stage LP. Scenarios for solving the FAM were generated as described in Section 4.1. The objective of scenario generation is to provide an estimation of the expected value, which is equivalent to numerical integration. As in Smith (2004) and Pilla et al. (2008), second-stage demand scenarios were generated using a truncated normal distribution with known mean and standard deviation values for each itinerary fare class, where Pilla et al. (2008) used a coefficient of variation (CV) of 0.05, and here we use a CV of 0.25. This higher CV requires a higher number of scenarios. Pilla et al. (2008) used a confidence level approach to identify an appropriate number of scenarios. Allowing an approximate 0.1% error on profit with 95% confidence, this approach recommended 60 scenarios. The number of scenarios for the L-shaped method is debatable, as it is problem-specific and depends on the convergence of the recourse function value. For this research it was taken as 60, the same as that which was required for the MARS approximation, but the L-shaped method might require more. Increasing the number of scenarios will increase the time required for obtaining the L-shaped solution. In addition, some numerical issues were encountered with the L-shaped method that caused infeasibility in the recourse problem. In order to maintain complete recourse, near integer numbers were rounded while generating solutions in the first stage.

The DoE in the DACE Phase was identical to Pilla et al. (2008). Using the average profit response values at the CCA DoE points, a MARS approximation was fit using an automatic stopping rule as discussed in ?. The fit resulted in 64 basis functions with a coefficient of determination (R^2) of 99.306%, which was comparable to the MARS results in Pilla et al. (2008). The MARS approximation was tested for its concavity by generating 1000 random points and checking the following inequality for each pair of points (x_H^1, x_H^2)

$$\hat{\theta}(x_H^1) \geq \hat{\theta}(x_H^2) + \nabla \hat{\theta}(x_H^2)(x_H^1 - x_H^2). \quad (51)$$

Since the first-stage assignment needs to be binary, MIP cuts were generated from the second stage and added to the first stage in both methods. The stopping criteria used was:

$$(\text{RMP objective} - 0.025 * \text{RMP objective}) \leq \text{Recourse function value} \quad (52)$$

The initial 10 - 15 MIP cuts were very effective, but later the cuts were less valuable as can be seen in Figure 5. In order to decrease the time required to solve the problem, we implemented two methods. The first method was to relax the first stage to allow fractional values, and LP cuts were added from the second stage. Once the criteria

$$(\text{RMP LP objective} - 0.001 * \text{RMP LP objective}) \leq \text{Recourse function value} \quad (53)$$

was met, MIP cuts were again added. Using this method the time required was reduced from days to hours for the L-shaped method and was only minutes when using MARS-CP. The second method was to increase the node limit in each iteration linearly, and once the stopping criteria for

the MIP objective was reached, the problem was solved without any node limit. Another criteria that can be used to speed up the process of solving the two-stage SP FAM is not solving the current master problem to optimality but instead to terminate it as soon as a feasible solution is produced that has a value below (upper bound - ϵ); thus this incumbent is an ϵ -optimal solution (Geoffrion and Graves 1974; Adams and Sherali 1993).

The results for the two methods are shown in Table 1, where the computation of the DACE Phase to generate the MARS recourse function is detailed in Table 2. The first-stage MIP assignment obtained using the two approaches was used to get the recourse function values, and they happen to be very similar (difference was within 0.1%). This can be attributed to the fact that the objective function of an SP problem is typically flat, giving rise to similar objective values but with different solutions, as mentioned in Survajeet and Higle (1996). Table 1 shows that there is a significant reduction of time in the optimization phase by using the MARS approximation as the recourse function. This can be attributed to the fact that the second-stage recourse evaluation is based upon a closed-form formula as opposed to solving linear programs. However, the trade-off is that the recourse function has to be generated earlier to do the optimization.

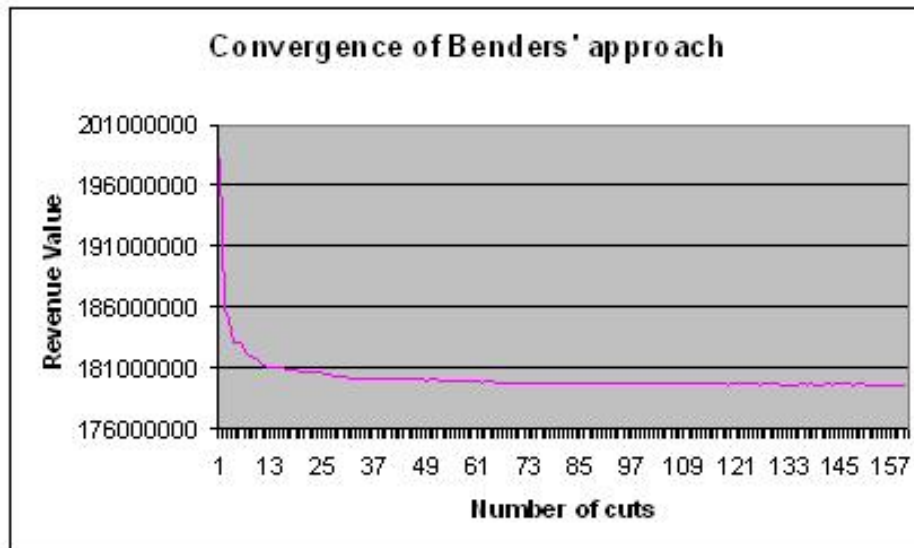


Figure 5: Convergence of the objective

The time required for generating the recourse function values can be decreased by considering fewer variables and fewer design points. Some directions for future work are mentioned in Section 6. Using the node limit method, the total time required to solve the two-stage SP FAM is around 5.80 days using our MARS-CP method compared to 8.68 days for the L-shaped method.

Table 1: Comparison of MARS and L-shaped methods by relaxing initially. CV = 0.25

	MARS-CP	L-shaped
Number of Scenarios	60	60
DACE Phase	5.80 days	N/A
Number of LP cuts	95	1346
Time Required for LP cuts	99.6 sec	8.65 days
LP Objective	176,989,661.49	176,943,047.80
Number of MIP cuts	0	0
Cumulative Time Required for MIP cuts	3.07 min	8.68 days
Total Computational Time	5.80 days	8.68 days
MIP objective	176,923,299.66	176,851,699.50
Recourse function value	176,197,819.53	176,069,660.19

Table 2: Steps to generate first-stage solution using MARS approximation. CV = 0.25

Step No.	Description	Time required
1	Dimensionality Reduction	1.5 hours
2	Generate LH design using MATLAB	4.5 hours
3	Map the LH design to points	0.32 hours
4	Check for feasibility	0.43 hours
5	Use the L1 norm for projection	0.97 hours
6	To find the proximate extreme point	1.1 hours
7	Generate interior points	5 min
8	Generate recourse function values	5.4 days
9	Fit a MARS approximation	0.75 hours
10	Optimize to obtain a first-stage solution	0.3 hours

In order to quantify the value of D^3 and the two-stage model, a single-stage FAM plus the PMM was constructed as shown below:

$$\max \kappa = \left[- \sum_{l \in L} \sum_{f \in F} C_{fl} x_{fl} + \sum_{i \in I} f_i z_i \right] \quad (54)$$

$$\text{s.t. } a_{v^-} + \sum_{l \in A_v} x_{fl} - \sum_{l \in D_v} x_{fl} - a_{v^+} = 0 \quad \forall v \in V, \quad (55)$$

$$\sum_{f \in F} x_{fl} = 1 \quad \forall l \in L, f \in F, \quad (56)$$

$$\sum_{o \in O_f} a_o + \sum_{l \in L_0} x_{fl} \leq M_f \quad \forall f \in F, \quad (57)$$

$$\sum_{i \in I} z_i \leq \sum_{f \in F} Cap_f x_{fl} \quad \forall l \in L, \quad (58)$$

$$0 \leq z_i \leq D_i \quad \forall i \in I, \quad (59)$$

$$x_{fl} \in \{0, 1\} \quad \forall l \in L, f \in F \quad (60)$$

$$a_{v^+} \geq 0 \quad \forall v \in V, \quad (61)$$

where κ is the objective value of the single-stage model. Let x_{ss}^* be the MIP solution to the single-stage model, x_{2sm}^* be the MIP solution to the two-stage model using MARS-CP, and x_{2sb}^* be the MIP solution to the two-stage model using the L-shaped method. Let $\text{Pfam}(X)$ and $\text{Prec}(X)$ be the profit from the single-stage FAM objective function and from the recourse function, respectively. Then the value of D^3 can be calculated as: $\text{Prec}(x_{ss}^*) - \text{Pfam}(x_{ss}^*)$, and the value of the two-stage model using MARS recourse function is: $\text{Prec}(x_{2sm}^*) - \text{Prec}(x_{ss}^*)$. Similarly the value of the two-stage model using L-shaped method is given as: $\text{Prec}(x_{2sb}^*) - \text{Prec}(x_{ss}^*)$.

For our airline example, Pfam and Prec values are shown in Table 3. The value of D^3 is

Table 3: Solutions.

	Profit
$\text{Pfam}(x_{ss}^*)$	\$164,389,021.49
$\text{Prec}(x_{ss}^*)$	\$175,471,734.38
$\text{Prec}(x_{2sm}^*)$	\$176,197,819.53
$\text{Prec}(x_{2sb}^*)$	\$176,069,660.19

$\$175,471,734.38 - \$164,389,021.49 = \$11,082,712.89$; which is a 6.74% improvement. The value of the two-stage SP FAM for MARS-CP is $\$176,197,819.53 - \$175,471,734.38 = \$726,085.15$ (a 0.41% increase), and the value of the two-stage SP FAM for the L-shaped method is $\$176,069,660.19 - \$175,471,734.38 = \$597,925.81$ (0.34% increase). From the result, the value of the two-stage SP FAM for MARS-CP is greater than the value of the two-stage SP FAM for the L-shaped method. However, these differences are within the tolerance of the stopping criteria of the computational experiments.

6. Conclusion

This paper demonstrates and develops the MARS-CP algorithm to optimize a two-stage SP FAM. In the first stage of the two-stage SP FAM, a crew compatible assignment is made, and the

D^3 concept of swapping aircraft within crew-compatible families to optimize revenue is achieved in the second phase. The DACE approach to solve the two-stage SP FAM is decomposed into two phases—the DACE Phase and the Optimization Phase. The DACE Phase, which was described in Pilla et al. (2008), involves estimating the expected recourse function for the two-stage SP FAM problem using a MARS approximation over a discretized first-stage decision space based on a Latin hypercube design. This paper completes this method with the development of the Optimization Phase in which we optimized the MARS approximation function using a cutting plane algorithm. Using the two-stage SP FAM application from Pilla et al. (2008), we demonstrated an improvement in computational effort over the L-shaped method . In future research, we will employ a multi-cut algorithm to solve the two-stage SP FAM problem. In addition, the computation required by our DACE-based approaches can be reduced via data mining techniques that reduce the dimension of the first-stage decision space.

References

- Abara, J., 1989. Applying integer linear programming to the fleet assignment problem. *Interface* 19, 20–28.
- Adams, W., Sherali, H., 1993. Mixed-integer bilinear programming problems. *Mathematical Programming* 59, 279–305.
- Barnhart, C., Boland, N., Clarke, L., Johnson, E., Nemhauser, G., Shenoi, R., 1998. Flight string models for aircraft fleet and routing. *Transportation Science* 32, 208–220.
- Barnhart, C., Kniker, T., Lohatepanont, M., 2002. Itinerary-based airline fleet assignment. *Transportation Science* 36, 199–217.
- Berge, M., Hopperstad, C., 1993. Demand driven dispatch: A method of dynamic aircraft capacity assignment models and algorithms. *Operations Research* 41, 153–168.
- Chen, V., 2001. Measuring the goodness of orthogonal array discretizations for stochastic programming and stochastic dynamic programming. *SIAM Journal of Optimization* 12, 322–344.
- Chen, V., Ruppert, D., Shoemaker, C., 1999. Applying experimental design and regression splines to high dimensional continuous-state stochastic dynamic programming. *Operations Research* 47, 38–53.
- Chen, V.C.P., Günther, D., Johnson, E.L., 2003. Solving for an optimal airline yield management policy via statistical learning. *Journal of the Royal Statistical Society, Series C* 52, 1–12.
- Chen, V.C.P., Tsui, K.L., Barton, R.R., Meckesheimer, M., 2006. Design, modeling, and applications of computer experiments. *IIE Transactions* 38, 273–291.
- Clarke, L., Hane, C., Johnson, E., Nemhauser, G., 1996. Maintenance and crew considerations in fleet assignment. *Transportation Science* 30, 249–260.
- Farkas, A., 1996. The Influence of Network Effects and Yield Management on Airline Assignment Decisions. Ph.D. thesis. Massachusetts Institute of Technology.
- Friedman, J.H., 1991. Multivariate adaptive regression splines. *The Annals of Statistics* 19, 1–141.
- Geoffrion, A.M., Graves, G., 1974. Multicommodity distribution system design by benders decomposition. *Management Science* 20, 822–844.
- Gu, Z., Johnson, E., Nemhauser, G., Wang, Y., 1994. Some properties of the fleet assignment problem. *Operations Research Letters* 15, 59–71.
- Hane, C., Barnhart, C., Johnson, E., Nemhauser, G., Sigismondi, G., 1995. The fleet assignment problem: Solving a large scale integer program. *Mathematical Programming* 70, 211–232.
- Jacobs, T., Johnson, E., Smith, B., 1999. O-d fam: Incorporating passenger flows into the fleet process, in: *AGIFORS Symposium Proceedings*, New Orleans, LA, USA.
- Kniker, T., 1998. Itinerary-Based Airline Fleet Assignment. Ph.D. thesis. Massachusetts Institute of Technology.

- Listes, O., Dekker, R., 2002. A scenario aggregation based approach for determining a robust airline fleet composition. Technical Report Econometric Institute Report 17. Erasmus University Rotterdam, The Netherlands.
- Pilla, V., Rosenberger, J., Chen, V., 2008. A statistical computer experiments approach to airline fleet assignment. *IIE Transactions* 40, 524–537.
- Rockafellar, R., Wets, R., 1991. Scenarios and policy aggregation in optimization under uncertainty. *Mathematics of Operations Research* 16, 119–147.
- Rosenberger, J., Johnson, E., Nemhauser, G., 2003. A robust fleet assignment model with hub isolation and short cycles. *Transportation Science* 38, 357–368.
- Rushmeier, R., Kontogiorgis, S., 1997. Advances in the optimization of airline fleet assignment. *Transportation Science* 31, 159–169.
- Savelsbergh, M., 1994. Preprocessing and probing techniques for mixed integer programming problems. *ORSA Journal on Computing* 6, 445–454.
- Sherali, H.D., Bish, E.K., Zhu, X., 2006. Airline fleet assignment concepts, models, and algorithms. *European Journal of Operational Research* 172, 1–30.
- Sherali, H.D., Zhu, X., 2008. Two-stage fleet assignment model considering stochastic passenger demands. *Operations Research* 56, 383–399.
- Smith, B., 2004. Robust Airline Fleet Assignment. Ph.D. thesis. Georgia Institute of Technology.
- Subramanian, R., Scheff, R., Quillinan, J., Wiper, D., Marsten, R., 1994. Coldstart: fleet assignment at delta airlines. *Interfaces* 24, 104–120.
- Survajeet, S., Hige, J., 1996. Stochastic Decomposition. Kluwer Academic Publishers, Boston, MA.
- Talluri, K.T., 1996. Swapping applications in a daily fleet assignment. *Transportation Science* 30, 237–248.
- Tsai, J., Chen, V., 2005. Flexible and robust implementations of multivariate adaptive regression splines within a wastewater treatment stochastic dynamic program. *Quality and Reliability Engineering International* 21, 689–699.