COSMOS Technical Report 11-02:
The Scalarization and Equivalence of Standard Optimization Criteria

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Abstract

We show that any standard optimization criterion (SOC) has a scalar equivalence, i.e., is equivalent to the maximization of a real-valued function. We also demonstrate that a scalar equivalence of any SOC can be solved as one of any other SOC. In summary, all solutions and only solutions to an optimization problem involving one SOC can be obtained in terms of any other. Thus all SOCs are directly equivalent. Finally, we consider optimization problems involving SOCs as formal decision problems and conjecture whether all decision problems can be solved by a real-valued maximization problem over a suitable feasible region.

Keywords:

Multiple criteria analysis, Multiple objective programming, Optimization criteria, Scalarization, Decision problems

1. Introduction

An optimization criterion defines the notion of the term "best" in making a "best" decision. In this study, a standard optimization criterion (SOC) refers to Pareto (including the scalar case), satisficing, maximin, and standard cone-ordered optimization, as well as the more general notion of standard set-valued optimization. We show here that all SOCs are scalarizable. All solutions and only solutions for any SOC problem can be obtained by the maximization of a real-valued objective function subject to certain constraints. No convexity or concavity assumptions are required either on the objective functions or feasible region. We further demonstrate that any SOC is equivalent to any other SOC; i.e., all solutions and only solutions to one SOC problem can be directly obtained as solutions to any other SOC. One type of optimization problem can be solved as any other. Finally, we note that many formal decision problems can be solved by real-valued maximizations. The question then arises whether all formal decision problems can be solved in such a manner.

2. Preliminaries

In this section we define the SOCs to be studied and present definitions, notation, and results to be used later.

2.1 Maximin Problem

Let $f : R^n \times R^m \to R$ be a real-valued function. For each $\mathbf{x} \in A \subset R^n$, define the set $B(\mathbf{x}) \subset R^m$ to be a nonempty feasible region. Assume that the function $g(x) = \min_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y})$ is well-defined for all $\mathbf{x} \in A$. The general maximin problem [3] can be stated as

$$\max_{\mathbf{x} \in A \subset R^n} \min_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y}).$$

Note that for different $\mathbf{x}_1, \mathbf{x}_2 \in A \subset R^n$, the associated feasible regions $B(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$ are not necessarily identical. In another words, this formulation restricts the feasible choices of $\mathbf{y}$ depending on the certain choices of $\mathbf{x}$. If $B(\mathbf{x}) = B$ for all $\mathbf{x} \in A \subset R^n$, the above problem takes the more familiar form

$$\max_{\mathbf{x} \in A \subset R^n} \min_{\mathbf{y} \in B \subset R^m} f(\mathbf{x}, \mathbf{y}).$$

In particular, if $B = \{1, ..., n\} \subset R$ for some given positive integer n, the problem becomes the standard discrete maximin problem

$$\max_{\mathbf{x} \in A \subset R^n} \min\{ f(\mathbf{x}, 1), ..., f(\mathbf{x}, n)\}.$$

Example 2.1.1 Let $A = [1,9] \subset R$ and $B(x) = \{y \in [1, x] : x - y \geq y - 1\}$ for each $x \in A$. Define $f(x, y) = \dfrac{x}{y}$ for $x \in A, y \in B(x)$, and consider the maximin problem

$$\max_{x \in [1,9]} \min_{y \in B(x)} \frac{x}{y}.$$

In this example, the feasible region of variable *y* in minimization depends on the value of variable *x* given. For example, we have that $B(5) = [1,3]$, while $B(7) = [1,4]$.

2.2 Pareto Optimization

Let $A \subset R^m$ be a set of feasible solutions and $f : R^m \to R^n$ be an *n*-dimensional objective function $f(\mathbf{x}) = (f_1(\mathbf{x}),..., f_n(\mathbf{x}))$ for all $\mathbf{x} \in A$, where $f_i : R^m \to R$ is defined to be the i^{th} objective function of the problem, $i = 1,..,n.$ Then Pareto maximization, or vector maximization, can be stated as

$$\operatorname*{Vmax}_{\mathbf{x} \in A} (f_1(\mathbf{x}),..., f_n(\mathbf{x})).$$

A feasible solution $\mathbf{x} \in A$ is called a Pareto maximum, or efficient point, if there is no $\mathbf{y} \in A$ such that $f_i(\mathbf{x}) \le f_i(\mathbf{y})$ for all $i = 1,..,m$ and if $f_j(\mathbf{x}) < f_j(\mathbf{y})$ at least one index *j*. The set $\{ f(\mathbf{x}) \in R^n : \mathbf{x} \text{ are Pareto maxima over A} \}$ is called the Pareto frontier, or efficient frontier, of the problem.

2.3 Goal Programming

Goal programming is usually written as a scalar maximization or minimization of a function involving only the deviational variables for the goal constraints. However, we present here the more general definition as given in [7] in which goal programming is formulated as an equivalent Pareto maximization.

Let $f_i : R^m \to R$ for $i = 1,..., n$ be the goal functions and $b_1,..., b_n$ represent the associated aspiration levels for objective 1 to *n*, respectively. Then the goal programming problem can be stated

$$\left\{ \begin{array}{cc} \min\limits_{\mathbf{x}, s^+, s^-} & (s_1^- \text{or } s_1^+,..., s_n^- \text{or } s_n^+) \\ \text{s.t.} & f_1(x) + s_1^- - s_1^+ = b_1 \\ & \vdots \\ & f_n(x) + s_n^- - s_n^+ = b_n \\ & s_i^- \cdot s_i^+ = 0 \\ & s_i^-, s_i^+ \ge 0, \mathbf{x} \in A \end{array} \right\}.$$

The objective is to minimize the deviations s_i^-, s_i^+ to obtain a feasible **x** making the goal functions as close to the aspiration levels b_i as possible. For more details, see [14] and [18].

2.4 Cone-ordered Maximization

Definition 2.4.1 A nonempty set $C \subset R^n$ is called a cone if and only if $\lambda \mathbf{c} \in C$ for all $\mathbf{c} \in C$ and $\lambda \geq 0.$ The cone C is pointed if the set $C \cap -C$ contains only the vector of zero. Moreover, a convex cone C is a cone such that $\lambda_1 \mathbf{c}_1 + \lambda_2 \mathbf{c}_2 \in C$ for all $\mathbf{c}_1, \mathbf{c}_2 \in C$ and $\lambda_1, \lambda_2 \geq 0.$

Example 2.4.2 An important pointed convex cone in R^n is the nonnegative orthant $R^n_{\geq} = \{(x_1, ..., x_n) : x_i \geq 0, i = 1, ..., n\},$ for which Pareto optimization is cone ordered.

Example 2.4.3 Another important cone is called the lexicographic cone [1] defining lexicographic optimization [15], where individual goals are ordered by priority so that any higher level goal preempts a lower level one. For example, in R^2, the lexicographic cone is defined as

$$L = \{(x, y) \in R^2 : x > 0, \text{ or else } x = 0 \text{ and } y > 0\}.$$

Notice that the lexicographic cone is a pointed and convex, but the line $\{(x, y) \in R^2 : x = 0, y < 0\}$ is missing.

Definition 2.4.4 Let C be a pointed convex cone in R^n and define a relation order $\leq_C$ on R^n as follows. For any $\mathbf{y}_1, \mathbf{y}_2 \in R^n$, we say that $\mathbf{y}_1 \leq_C \mathbf{y}_2$ if $\mathbf{y}_2 - \mathbf{y}_1 \in C.$ Define $\mathbf{y}_1 <_C \mathbf{y}_2$ if $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. In particular, we say that $\mathbf{y}_2$ dominates $\mathbf{y}_1$ if $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. A vector $\mathbf{y}_1 \in B \subset R^n$ is said to be non-dominated in B if there is no $\mathbf{y}_2 \in B$ such that $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. Denote the set $\max_C B$ as the set containing all non-dominated vectors in B with respect to the cone C.

Example 2.4.5 For the lexicographic cone of Example 2.4.3, we construct the order induced by it. Let $B = \{(0,0), (0,1), (1,0), (1,1)\} \subset R^2$ and L be the lexicographic cone in R^2. Then

$$(0,0) \leq_L (0,1), (0,1) \leq_L (1,0), \text{and} (1,0) \leq_L (1,1).$$

Definition 2.4.6 Let C be a pointed cone in R^n. A linear functional l is a function mapping R^n into R, which satisfies the following property:

$$l(\alpha_1\mathbf{y}_1+\alpha_2\mathbf{y}_2)=\alpha_1 l(\mathbf{y}_1)+\alpha_2 l(\mathbf{y}_2) \text{ for all } \alpha_1,\alpha_2\in R \text{ and } \mathbf{y}_1,\mathbf{y}_2\in R^n.$$

A linear functional l is said to be strictly positive on C if $l(\mathbf{c})>0$ for all non-zero vectors $\mathbf{c}\in C$. The dual cone associated with C is the collection of all strictly positive linear functionals l on C and denoted by $C^+=\{l : l(\mathbf{c})>0,\ \mathbf{c}\in C, \mathbf{c}\neq\mathbf{0}\}$.

Example 2.4.7 Consider R^2 with the order induced by the nonnegative orthant cone $R^2_{\geq}=\{(x,y): x,y\geq 0\}$. We construct a linear functional $l: R^2\to R$ given by $l(x,y)=x+y$ for all $x,y\in R.$ Then l is a linear functional such that $l(x,y)=x+y>0$ for all non-zero $(x,y)\in R^2_{\geq}$, so $(R^2_{\geq})^+\neq\phi$.

The following standard results are used. In particular, note that the pointedness of a cone is required for a strictly positive linear functional on C to exist.

Lemma 2.4.8 Let C be a pointed cone in R^n and assume that $C^+\neq\phi.$ If $\mathbf{x}_0<_C\mathbf{x}_1$ then $l(\mathbf{x}_0)<l(\mathbf{x}_1)$ for any $l\in C^+$.

Theorem 2.4.9 (cone separation theorem [2]). Let S_1,S_2 be closed convex cones in R^n such that $S_1\cap S_2=\{\mathbf{0}\}$, and denote the topological dual of R^n by $(R^n)'$. Suppose that S_1^+ has nonempty interior in some topology τ which provides R^n as the dual of $(R^n)'$. Then there exists $s^+\in(S_1^+)^0$ such that $-s^+\in S_1{}^+$ and $s^+(\mathbf{s}_1)>0$ for all non-zero vector $\mathbf{s}_1\in S_1$.

Corollary 2.4.10 [6] If $C^+\neq\phi$, then C is a pointed cone.

It follows from Corollary 2.4.10 that pointedness is a necessary condition for existence of a strictly linear functional on C.

Corollary 2.4.11[4] $L^+=\phi$ where L is a lexicographic cone in R^n.

For the remainder of this paper we consider cone-ordered maximization only for a pointed convex cone C with $C^+\neq\phi$ with notable exception of the lexicographic cone L.

Definition 2.4.12 Let C be a pointed convex cone in R^n with $C^+ \neq \phi$ and $f: R^m \to R^n$. Suppose $A \subset R^m$ is a feasible region. Then cone-ordered maximization, or C-maximization, can be written as

$$C\max_{\mathbf{x}\in A} f(\mathbf{x})$$

The problem is to find all $\mathbf{x} \in A$ for which $f(\mathbf{x}) \in \max_C f(A)$, for $f(A) = \bigcup_{\mathbf{x}\in A} f(\mathbf{x})$ and $\max_C f(A) = \{\text{All non-dominated } f(\mathbf{x}) \text{ in } R^n \text{ for } \mathbf{x} \in A\}$, i.e., to find non-dominated $f(\mathbf{x})$ for all feasible solution $\mathbf{x} \in A$. General optimality conditions are found in [9].

If a cone C is $R^n_{\geq} = \left\{(c_1,\dots,c_n) : c_i \geq 0, i = 1,\dots,n\right\}$, C-maximization becomes Pareto maximization with n objective functions.

2.5 Lexicographic Maximization

Definition 2.5.1 Let $L_1 = \{(x_1,\dots,x_n) \in R^n : x_1 > 0\}$ and $L_j = \{(x_1,\dots,x_n) \in R^n : x_1 = 0, x_{j-1} = 0, x_j > 0\}$ for $j = 2,\dots,n$. The cone $L = [\bigcup_{j=1,\dots,n} L_j] \bigcup \{\mathbf{0}\}$ is called the lexicographic cone in R^n. Then cone-ordered maximization with respect L becomes the lexicographic maximization

$$\underset{\mathbf{x}\in B\subset R^m}{\text{Leximax}}(f_1(\mathbf{x}),\dots,f_n), \text{where } f_i : R^m \to R, i = 1,\dots,n.$$

The problem now is to find a feasible solution $\mathbf{x} \in B$ for which there is no other vector $\mathbf{y} \in B$ such that $\mathbf{x} <_L \mathbf{y}$.

Example 2.5.2 In Example 2.4.5, the set $B = \left\{(0,0),(0,1),(1,0),(1,1)\right\} \subset R^2$. Notice that (1,1) is the only non-dominated vector in B and therefore the solution to the lexicographic maximization.

2.5.3 Scalarization for lexicographic maximization

The nature of the lexicographic order allows us to construct a scalarization even though the dual cone $L^+ = \phi$ as noted in Corollary 2.4.11.We illustrate in R^3. The general case is similar.

Consider the lexicographic maximization

$$\left\{\begin{matrix} \underset{\mathbf{x}}{\text{Leximax}} & (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \subset R^n \end{matrix}\right\} \text{ where } f_i : R^n \to R \text{ for } i = 1,2,3.$$

This problem can be solved in stages corresponding to the objective functions.

Step 1: Solve $\max_{\mathbf{x}\in A} f_1(\mathbf{x})$ and denote f_1^* the optimal objective value of this problem.

Step 2: Solve $\left\{\begin{array}{ll} \max_{\mathbf{x}} & f_2(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) = f_1^* \\ & \mathbf{x}\in A \end{array}\right\}$ and denote f_2^* the optimal objective value of this problem.

Step 3: Solve $\left\{\begin{array}{ll} \max_{\mathbf{x}} & f_3(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) = f_1^* \\ & f_2(\mathbf{x}) = f_2^* \\ & \mathbf{x}\in A \end{array}\right\}.$

Solutions from the scalar problem in Step 3 are solutions for the given lexicographic maximization and vice versa. Thus the maximization problem in Step 3 is a scalar equivalence for the given lexicographic maximization. Details about a more general lexicographic problem can be found in [7].

2.6 Set-Valued Optimization

Definition 2.6.1 Let $F: R^m \to 2^{R^n}$ be a point-to-set map and C a pointed convex cone in R^n. The problem $\max_{\mathbf{x}\in A} F(\mathbf{x})$ is a set-valued maximization to find all feasible vector $\mathbf{x}\in A \subset R^m$ such that $F(\mathbf{x}) \cap \max_C F(A) \neq \phi,$ where $F(A) = \bigcup_{\mathbf{x}\in A} F(\mathbf{x}),$ i.e., of finding all feasible $\mathbf{x}$ for which there exists $\mathbf{y}\in F(\mathbf{x})$ and $\mathbf{y}\in \max_C F(A).$ Set-valued was defined in [11], where general optimality conditions were given, and generalizes cone-ordered maximization. We consider a set-valued maximization with a pointed convex cone for which $C^+ \neq \phi.$

Example 2.6.2 Let $A = \{(x_1, x_2)\in R^2 : x_1 + x_2 \le 1, x_1, x_2 \ge 0\} \subset R^2,$ and $C = R^2_{\geq}.$ Define $F(x_1, x_2) = [0, x_1]\times[0, x_2] \subset R^2$ for all $x_1, x_2 \in [0,1].$ Notice that the function F is a point-to-set map, and the problem $\max_{\mathbf{x}\in A} F(\mathbf{x})$ is a set-valued maximization. The set of solutions the set $\{(x_1, x_2)\in R^2 : x_1 + x_2 = 1, x_1, x_2 \ge 0\}.$

3. Scalar Equivalence of SOCs

We now show that any SOC optimization problem has a scalar equivalence. For example, a multiple-objective optimization problem is typically solved by transforming the original problem into the scalar maximization of a real-valued function in which certain parameters are varied to give alternate solutions to the original multiple-objective problem. See [7], [14], [15], [22], and [23].

However, the most frequently used such scalarizations of Pareto optimization require assumptions about the convexity or concavity of functions to guarantee that a scalarization exists and yields all solutions to the original Pareto problem. Because of this limitation, we say that a non-scalar optimization problem is scalarizable if and only if all solutions and only solutions of the non-scalar problem can be obtained by a possibly parameterized scalar maximization problem called its equivalent scalarization. In that case, the scalarization is said to be scalar equivalent to the original non-scalar problem.

The notion of scalar equivalence stems then work of Corley [8] (see also [7] and [15]) in cone-ordered optimization, which includes Pareto and scalar optimization. This scalar equivalence involves no more effort to solve than scalarizations requiring various convexity or concavity assumptions on the original problem. It is now known as a hybrid method [7] from its relation to the Corley hybrid fixed point theorems of [10].

3.1 Maximin

We denote $A1$ below as a given maximin problem, where $g(\mathbf{x}) = \min_{\mathbf{y}\in B(\mathbf{x})\subset R^m} f(\mathbf{x},\mathbf{y})$ for all $\mathbf{x}\in A$.

The problem $A2$ is an obvious equivalence of $A1$ after introducing a real-value decision variable v to be the value of $g(\mathbf{x})$. We prove that $A3$ is a scalar equivalence of the given maximin $A1$. We note that in $A3$ the variable $\mathbf{y}$ in the set of constraints is not a decision variable but is a parameter relating the constraints of $A2$ to the set $B(\mathbf{x})$ for each feasible point $\mathbf{x}$.

$$A1:\ \max_{\mathbf{x}\in A\subset R^n} g(\mathbf{x}) \qquad A2:\begin{Bmatrix} \max_{\mathbf{x},v} & v \\ \text{s.t.} & v = g(\mathbf{x}) \\ & \mathbf{x}\in A\subset R^n, v\in R \end{Bmatrix} \qquad A3:\begin{Bmatrix} \max_{\mathbf{x},v} & v \\ \text{s.t.} & v \le f(\mathbf{x},\mathbf{y}), \forall \mathbf{y}\in B(\mathbf{x}) \\ & \mathbf{x}\in A\subset R^n, v\in R \end{Bmatrix}$$

Lemma 3.2.1 If $(v_3^*,\mathbf{x}_3^*)$ is a solution to $A3$, then $v_3{*} = f(\mathbf{x}_3{*},\mathbf{y}{*})$ for some $\mathbf{y}{*}\in B(\mathbf{x}_3^*)$. Thus $v_3{*} = g(\mathbf{x}_3{*})$ and $(v_3^*,\mathbf{x}_3^*)$ is also a feasible solution to $A2$.

Proof. Assume that $(v_3^*,\mathbf{x}_3^*)$ is a solution to $A3$. With the feasibility, we observe that $v_3^* \le f(\mathbf{x}_3^*,\mathbf{y})$ for all $\mathbf{y}\in B(\mathbf{x}_3^*)$. To obtain a contradiction, suppose $v_3^* < f(\mathbf{x}_3^*,\mathbf{y})$ for all $\mathbf{y}\in B(\mathbf{x}_3^*)$. By the assumption that $g(\mathbf{x}_3^*)$ exists, we have that $g(\mathbf{x}_3^*) = \min_{\bar{\mathbf{y}} \in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})$ is a finite real number. Then it follows that $\dfrac{v_3^* + \min_{\bar{\mathbf{y}} \in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2} \le f(\mathbf{x}_3^*, \mathbf{y})$ for all $\mathbf{y} \in B(\mathbf{x}_3^*)$, which implies that $(\dfrac{v_3^* + \min_{\mathbf{y} \in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2}, \mathbf{x}_3^*)$ is a feasible solution of $A3$. However, we also have that $v_3^* < \dfrac{v_3^* + \min_{\mathbf{y} \in B} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2}$, contradicting that v_3^* is the optimal objective value of $A3$. Thus, we can conclude that $v_3^* = f(\mathbf{x}_3^*, \mathbf{y}^*)$ for some $\mathbf{y}^* \in B(\mathbf{x}_3^*)$ and $v_3^* \le f(\mathbf{x}_3^*, \mathbf{y})$ for $\mathbf{y} \ne \mathbf{y}^*$, i.e., $v_3^* = g(\mathbf{x}_3^*)$. ■

Theorem 3.2.2 The point $(v^*, \mathbf{x}^*)$ is a solution to $A2$ if and only if $(v^*, \mathbf{x}^*)$ is a solution to $A3$.

Proof. Suppose $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. By the definition of the function g, we have that $(v^*, \mathbf{x}^*)$ is a feasible solution to $A3$ as well. To obtain a contradiction, suppose that $(v^*, \mathbf{x}^*)$ is not an optimal solution to $A3$. Then there is another feasible solution $(v_3^*, \mathbf{x}_3^*)$ of $A3$ such that $v^* < v_3^* \le f(\mathbf{x}_3^*, \mathbf{y}), \forall \mathbf{y} \in B(\mathbf{x}_3^*)$.

Case 1: $v_3^* = f(\mathbf{x}_3^*, \mathbf{y}^*)$ for some $\mathbf{y}^* \in B(\mathbf{x}_3^*)$. In this case $v_3^* = g(\mathbf{x}_3^*)$, and hence $(v_3^*, \mathbf{x}_3^*)$ is a feasible solution to $A2$. However, we have that $v^* < v_3^*$, contradicting that $(v_2^*, \mathbf{x}_2^*)$ is an optimal solution to $A2$.

Case 2: $v_3^* < f(\mathbf{x}_3^*, \mathbf{y}), \forall \mathbf{y} \in B(\mathbf{x}_3^*)$. Since $g(\mathbf{x}) = \min_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y})$ is well-defined for all $\mathbf{x} \in R^n$, let $\hat{v} = g(\mathbf{x}_3^*)$. By the construction, we have that $(\hat{v}, \mathbf{x}_3^*)$ is a feasible solution to $A2$. However, we also obtain the condition $v^* < v_3^* < \hat{v}$, contradicting that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. Thus we conclude that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$.

To establish the reverse implication, suppose $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$. By Lemma 3.2.1, we have that $(v^*, \mathbf{x}^*)$ is a feasible solution to $A2$. To obtain a contradiction, suppose that $(v^*, \mathbf{x}^*)$ is not an optimal to $A2$. Then there is another feasible solution $(v_2^*, \mathbf{x}_2^*)$ of $A2$ such that $v^* < v_2^*$. Since the feasible region of $A2$ is a subset of the feasible region of $A3$, it follows that

$(v_2^*, \mathbf{x}_2^*)$ is a feasible solution to $A3$ such that $v^* < v_2^*$. This inequality is a contradiction because $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$. Thus we obtain that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. ■

The next two corollaries follow immediately.

Corollary 3.2.3 For $f : A \times B \to R$, an equivalent scalarization for the maximin problem $\max_{\mathbf{x} \in A \subset R^n} \min_{\mathbf{y} \in B \subset R^m} f(\mathbf{x}, \mathbf{y})$ is

$$\left\{ \begin{array}{ll} \max\limits_{\mathbf{x}, v} & v \\ \text{s.t.} & v \le f(\mathbf{x}, \mathbf{y}), \forall \mathbf{y} \in B \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\}.$$

Corollary 3.2.4 For $f_i : A \to R$ for all $i \in 1, \ldots, n$ for a fixed positive integer n, an equivalent scalarization for the discrete maximin problem, $\max_{\mathbf{x} \in A \subset R^n} \min_{i \in \{1,\ldots,n\}} \{ f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}) \}$ is

$$\left\{ \begin{array}{ll} \max\limits_{\mathbf{x}, v} & v \\ \text{s.t.} & v \le f_1(\mathbf{x}) \\ & \vdots \\ & v \le f_n(\mathbf{x}) \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\}.$$

It should be noted that the scalar equivalence for the discrete maximin of Corollary 3.2.4 has been used extensively in [5], [15], and [26], for example, with at most a reference to Dantzig [12] for the linear case.

Example 3.2.5 Consider the following maximin problem.

$$\max_{x \in R} \min \{ f_1(x) = x, f_2(x) = -x \}$$

Algorithms for solving the problem with a discontinuous objective function have been developed in [13], [26], and [27]). However, the following of Corollary 3.2.4 indicates that $x^* = 0$

$$\left\{ \begin{array}{ll} \max\limits_{v, x} & v \\ \text{s.t.} & v \le x \\ & v \le -x \\ & v, x \in R \end{array} \right\}.$$

3.3 Pareto Maximization

For Pareto optimization with *m*-dimensional objective functions, where *m* is a positive integer, Corley [8] provided a scalar equivalence to the problem without assumptions such as convexity or concavity. It is discussed in [7] and [15]. This scalarization yields all solutions and only solutions for a given Pareto problem by solving a family of parameterized scalar problems. The scalar equivalence is stated as follows.

$$B(\mathbf{y}):\left\{\begin{array}{ll}\max\limits_{\mathbf{x}\in A\subset R^n} & \lambda\cdot(f_1(\mathbf{x}),\ldots,f_m(\mathbf{x}))\\ \text{s.t.} & (f_1(\mathbf{x}),\ldots,f_m(\mathbf{x}))-\mathbf{y}\in C\end{array}\right\}$$ for all $\mathbf{y}\in R^m$, where C is a pointed convex cone in R^m,

and $\lambda\in C^+ = \{\lambda\in R^m : \lambda\cdot\mathbf{c}>0, \forall\mathbf{c}\in C\setminus\{\mathbf{0}\}\}$ for given positive integers *n*, and *m*.

Example 3.3.1 Consider the following Pareto problem

$$\left\{\begin{array}{ll}\operatorname*{Vmax}\limits_{x_1,x_2} & (x_1,x_2)\\ \text{s.t.} & x_1+x_2\le 1\\ & x_1,x_2\ge 0\end{array}\right\}.$$

The above scalarization $B(\mathbf{y})$ in this problem becomes

$$B(y_1,y_2):\left\{\begin{array}{ll}\max\limits_{x_1,x_2} & x_1+x_2\\ \text{s.t.} & x_1\ge y_1\\ & x_2\ge y_2\\ & x_1+x_2\le 1\\ & x_1,x_2\ge 0\end{array}\right\},\ \text{for all}\ y_1,y_2\in R.$$

To illustrate the parameterization, choose $y_1=\frac{1}{2}$, and $y_2=\frac{1}{2}$. Then solving the problem $P(\frac{1}{2},\frac{1}{2})$ gives $(x^*,y^*)=(\frac{1}{2},\frac{1}{2})$. In theory we can similarly obtain all solutions of the Pareto problem, by solving $P(y_1,y_2)$ for all feasible choices of y_1 and y_2. In practice, a reasonable number of such solutions will approximate the Pareto frontier.

3.4 Standard Cone-ordered Maximization

Let C be a pointed convex cone in R^n with $C^+\ne\phi$. An equivalent scalarization for the cone-ordered maximization $C1$ below is $C2$.

$$\text{C1: } C\max_{\mathbf{x}\in A\subset R^n} f(\mathbf{x}) \qquad C2(\mathbf{w})\text{: } \left\{\begin{array}{ll} \max\limits_{\mathbf{x}} & l(f(\mathbf{x})) \\ \text{s.t.} & f(\mathbf{x})-\mathbf{w}\in C \\ & \mathbf{x}\in A\subset R^n \end{array}\right\}, \text{where } l\in C^+ \text{ for all } \mathbf{w}\in R^n$$

Theorem 3.4.1 If $\mathbf{x}_1$ is a solution of $C1$, then $\mathbf{x}_1$ solves $C2(\mathbf{w})$ for $\mathbf{w}=f(\mathbf{x}_1)$.

Proof. Assume that $\mathbf{x}_1$ solves $C1$. By the choice $\mathbf{w}=f(\mathbf{x}_1)$, we know that $\mathbf{x}_1$ is a feasible solution to $C2(f(\mathbf{x}_1))$. Let $\mathbf{x}_2$ be any feasible solution to $C2(f(\mathbf{x}_1))$. We therefore have $f(\mathbf{x}_2)-f(\mathbf{x}_1)\in C$. Since $\mathbf{x}_1$ solves $C1$, the only possibility is that $f(\mathbf{x}_2)=f(\mathbf{x}_1)$, so every feasible point of $C2(f(\mathbf{x}_1))$ is a solution as well. Since $\mathbf{x}_1$ is a feasible to $C2(f(\mathbf{x}_1))$, it solves $C2(f(\mathbf{x}_1))$. ■

Theorem 3.4.2 If $\mathbf{x}_2$ solves $C2(\mathbf{w})$ for $\mathbf{w}\in R^n$, then $\mathbf{x}_2$ is a solution of $C1$.

Proof. Assume that $\mathbf{x}_2$ solves $C2$ for some $\mathbf{w}$. To obtain a contradiction, suppose that $\mathbf{x}_2$ does not solve $C1$. Then there exists $\mathbf{x}_1\in A$ such that $f(\mathbf{x}_2)<_C f(\mathbf{x}_1)$, i.e., $f(\mathbf{x}_1)-f(\mathbf{x}_2)\in C\setminus\{\mathbf{0}\}$. It follows that $\mathbf{x}_1$ is a feasible solution of $C2(\mathbf{w})$. Since l is a strictly positive linear functional on C, we have $l(f(\mathbf{x}_1)-f(\mathbf{x}_2))>0.$ The linearity of l now yields that
$l(f(\mathbf{x}_1))-l(f(\mathbf{x}_2))=l(f(\mathbf{x}_1)-f(\mathbf{x}_2))>0.$ Thus $l(f(\mathbf{x}_1))>l(f(\mathbf{x}_2))$ in contradiction to the optimality of $\mathbf{x}_2$. ■

We note that the dual cone $L^+=\phi$ for the lexicographic cone L in R^n. Thus we cannot use Theorems 3.4.1 and 3.4.2 to construct an equivalent scalarization for lexicographic optimization. However, lexicographic maximization can be scalarizable with an alternate approach.

3.5 Standard Set-Valued Maximization

Denote the standard set-valued maximization below as $D1$. A scalar equivalence is presented in $D2$ for a convex, pointed cone $C\subset R^n$ for which $C^+\neq\phi$.

$$D1\text{: } \max_{\mathbf{x}\in A\subset R^m} F(\mathbf{x}) \qquad D2(\mathbf{w})\text{: } \left\{\begin{array}{ll} \max\limits_{\mathbf{x},\mathbf{t}} & l(\mathbf{t}) \\ \text{s.t.} & \mathbf{t}\in F(\mathbf{x}) \\ & \mathbf{t}-\mathbf{w}\in C \\ & \mathbf{x}\in A, \mathbf{t}\in R^n \end{array}\right\} \text{ for } l\in C^+ \text{and all } \mathbf{w}\in R^n.$$

An alternate scalarization for set-valued maximization has been proposed in [17]. However, the approach there requires assumptions involving convexity and concavity. A further scalarization is found in [20], but it gives only certain solutions.

Lemma 3.5.1 If the problem $D2(\mathbf{w})$ has a solution for some $\mathbf{w} \in R^n$, the problem $D1$ has a solution as well.

Proof. Suppose the problem $D2(\mathbf{w})$, where $\mathbf{w} \in R^n$, has a solution. Let $(\mathbf{x}_2, \mathbf{t}_2)$ be a solution of $D2(\mathbf{w})$. By feasibility, we have $\mathbf{t}_2 \in F(\mathbf{x}_2)$ and $\mathbf{w} \leq_C \mathbf{t}_2$. To obtain a contradiction, suppose that the set $\max F(A)$ is an empty set. Then there exists $\mathbf{x}_1 \in A$ and $\mathbf{t}_1 \in F(\mathbf{x}_1)$ for which $\mathbf{t}_2 <_C \mathbf{t}_1$, otherwise $\mathbf{t}_2 \in \max F(A)$. From the convexity of C, we have that $\mathbf{w} \leq_C \mathbf{t}_2$ and $\mathbf{t}_2 <_C \mathbf{t}_1$ implies $\mathbf{w} \leq_C \mathbf{t}_1$, so $(\mathbf{x}_1, \mathbf{t}_1)$ is feasible to $B2(\mathbf{w})$. However, since $\mathbf{t}_2 <_C \mathbf{t}_1$, by Lemma 2.4.8 we have $l(\mathbf{t}_2) < l(\mathbf{t}_1)$ in contradiction to the optimality of $(\mathbf{x}_2, \mathbf{t}_2)$. ■

Theorem 3.5.2 If $\mathbf{x}_1$ solves $D1$, then $(\mathbf{x}_1, \mathbf{t}_1)$ solves $D2(\mathbf{w})$ for $\mathbf{w} = \mathbf{t}_1 \in F(\mathbf{x}_1) \bigcap \max F(A)$.

Proof. Assume that $\mathbf{x}_1$ solves $D1$. Then there exists $\mathbf{t}_1 \in F(\mathbf{x}_1) \bigcap \max_C F(A)$, and $(\mathbf{x}_1, \mathbf{t}_1)$ is a feasible solution of $D2(\mathbf{t}_1)$. Now let $(\mathbf{x}_2, \mathbf{t}_2)$ be any feasible solution to $D2(\mathbf{t}_1)$. Therefore it follows that $\mathbf{t}_2 \in F(\mathbf{x}_2) \subset F(A)$ and $\mathbf{t}_2 - \mathbf{t}_1 \in C$. However, this conclusion is a contradiction to $\mathbf{t}_1 \in \max_C F(A)$ unless $\mathbf{t}_2 = \mathbf{t}_1$. Thus every feasible solution of $D2(\mathbf{t}_1)$ is also a solution. Since $(\mathbf{x}_1, \mathbf{t}_1)$ is a feasible solution of $D2(\mathbf{t}_1)$, then, it solves $D2(\mathbf{t}_1)$. ■

Theorem 3.5.3 If $(\mathbf{x}_2, \mathbf{t}_2)$ solves $D2(\mathbf{w})$ for $\mathbf{w} \in R^n$, then $\mathbf{x}_2$ is a solution of $D1$.

Proof. Assume that $(\mathbf{x}_2, \mathbf{t}_2)$ solves $D2(\mathbf{w})$ for $\mathbf{w} \in R^n$. To obtain a contradiction, suppose that $\mathbf{x}_2$ does not solve $D1$, i.e., $F(\mathbf{x}_2) \bigcap \max F(A) = \phi$. By Lemma 3.5.1, there exist a solution $\mathbf{x}_1$ of $D1$ and a vector $\mathbf{t}_1 \in F(\mathbf{x}_1)$ such that $\mathbf{t}_1 - \mathbf{t}_2 \in C \setminus \{\mathbf{0}\}$. Since $(\mathbf{x}_2, \mathbf{t}_2)$ is feasible to $D2(\mathbf{w})$, we have $\mathbf{t}_2 - \mathbf{w} \in C$. It follows that $\mathbf{t}_1 - \mathbf{w} \in C$ because of the convexity of C, so $(\mathbf{x}_1, \mathbf{t}_1)$ is feasible to $D2(\mathbf{w})$. However, by Lemma 2.4.8, $l(\mathbf{t}_2) < l(\mathbf{t}_1)$ in contradiction to the optimality of $(\mathbf{x}_2, \mathbf{t}_2)$. ■

Example 3.5.4 Recall the set-valued maximization problem in Example 2.6.2 with the problem

$$\max_{\mathbf{x}\in A} F(\mathbf{x}), \text{ where } F(x_1,x_2)=[0,x_1]\times[0,x_2]\subset R^2 \text{ for } x_1,x_2\in[0,1],$$

$$A=\{(x_1,x_2): x_1+x_2\le 1, x_1,x_2\ge 0\}\subset R^2, \text{ and } C=R^2_{\ge}.$$

The equivalent scalarization for this problem is

$$D(w_1,w_2):\left\{\begin{array}{ll}\max\limits_{x_1,x_2,t_1,t_2} & l(\mathbf{t})=t_1+t_2\\ \text{s.t.} & (t_1,t_2)\in F(x_1,x_2)\\ & t_1\ge w_1\\ & t_2\ge w_2\\ & x_1+x_2\le 1\\ & t_1,t_2,x_1,x_2\in R\end{array}\right\} \text{ for all } w_1,w_2\in R.$$

In order to obtain all solutions and only solutions of the set-valued maximization, we can theoretically solve the problem $D(w_1,w_2)$ for all feasible choices of w_1,w_2. For $w_1=\frac{1}{3}, w_2=\frac{2}{3}$, the problem $P(\frac{1}{3},\frac{2}{3})$ gives that $(x_1^*=\frac{1}{3}, x_2^*=\frac{2}{3}, t_1^*=\frac{1}{3}, t_2^*=\frac{2}{3})$ is a solution for the set-valued problem. Again, a large number of such solutions can approximate the Pareto frontier.

In Figure 3.3 we summarize our results that maximin problems, Pareto maximization, cone-ordered maximization, and set-valued maximization have equivalent scalarizations.

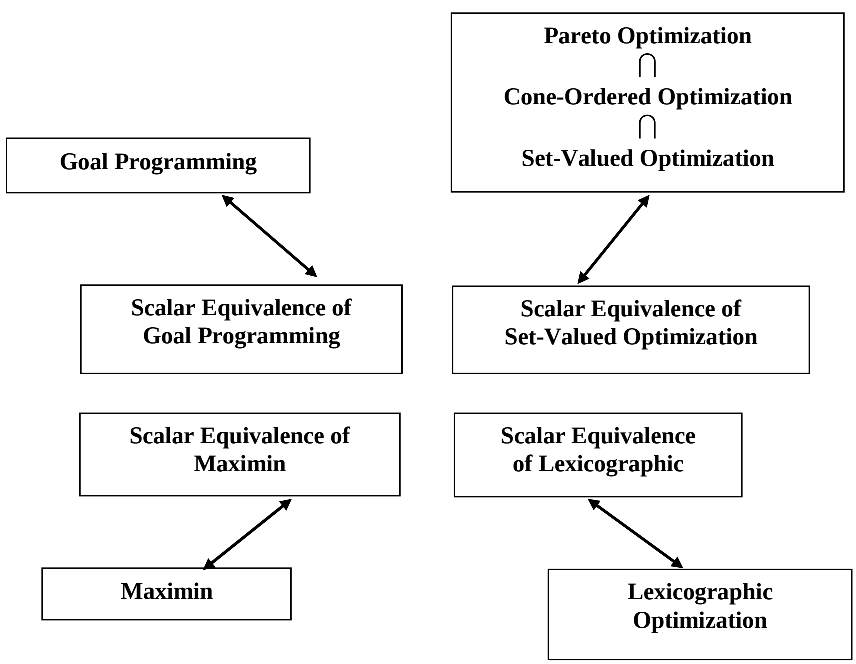

Figure 3.3 Scalar equivalence diagram.

4. Equivalences of SOCs

Let A, B be any SOC of Section 3. For example, suppose A is the Pareto maximization criterion and B is the maximin. We establish equivalences between SOC type A and SOC type B.

4.1 Criteria Equivalences of SOCs

Any two SOC A and SOC B are said to be criteria equivalent if and only the following conditions hold: (1) Given any problem involving SOC A, all solutions and only solutions can be obtained by solving a problem involving SOC B. (2) Given any problem involving SOC B, all solutions and only solutions can be obtained by solving a problem involving SOC A.

Lemma 4.1.1 Any SOC is criteria equivalent to Pareto.

Proof. The lemma is proved only for the maximin and Pareto case. The other cases are similar and established in [6].

Maximin as Pareto Maximization: Let $E1$ denote a given maximin problem, where $g(\mathbf{x}) = \min\{f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x})\}$ for all $\mathbf{x} \in R^m$ and $f_i : R^m \to R$ for all $i = 1,...,n$ for fixed n.

$$E1:\ \max_{\mathbf{x}\in A\subset R^m} g(\mathbf{x}) \qquad E2: \left\{ \begin{array}{ll} \max\limits_{v,\mathbf{x}} & v \\ \text{s.t.} & v = g(\mathbf{x}) \\ & \mathbf{x}\in A, v \in R \end{array} \right\} \qquad E3: \left\{ \begin{array}{ll} \operatorname{Vmax}\limits_{\mathbf{x},v} & (v, v, ..., v) \\ \text{s.t.} & v \le f_1(\mathbf{x}) \\ & \vdots \\ & v \le f_n(\mathbf{x}) \\ & \mathbf{x}\in A, v \in R \end{array} \right\}$$

The problem $E2$ is obviously equivalent to $E1$. Moreover, $E3$ is obviously equivalent to $E2$ because the objective function of $E3$ is just a replication of the objective function of $E2$. Obviously any single optimization of a real-valued function can be transformed to an equivalent Pareto optimization in this way.

Pareto Maximization as Maximin: Consider the following problems $F1$ and $F2$:

$$F1:\ \operatorname{Vmax}_{\mathbf{x}\in A\subset R^m} \ (f_1(\mathbf{x}), \cdots, f_n(\mathbf{x})) \qquad F2: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x}} & f_1(\mathbf{x}) + ... + f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) \ge y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \ge y_n \\ & \mathbf{x}\in A \end{array} \right\}.$$

for all $y_1, y_2, ..., y_n \in R$.

$F1$ is a given Pareto maximization problem, and the problem $F2$ represents an equivalent scalarization as in [8]. Consider now the maximin equivalence $F3$ of $F2$

$$F3:\left\{\begin{array}{lc} \max\limits_{\mathbf{x}} \min & \left\{\sum_{j=1}^{n} f_j(\mathbf{x}), \sum_{j=1}^{n} f_j(\mathbf{x})+1, \ldots, \sum_{j=1}^{n} f_j(\mathbf{x})+(n-1)\right\} \\ \text{ s.t. } & f_1(\mathbf{x}) \geq y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \geq y_n \\ & \mathbf{x} \in A \end{array}\right\} \text{ for all } y_1, y_2, \ldots, y_n \in R.$$

Since the value of $\min\left\{\sum_{j=1}^{n} f_j(\mathbf{x}), \sum_{j=1}^{n} f_j(\mathbf{x})+2, \ldots, \sum_{j=1}^{n} f_j(\mathbf{x})+n\right\}=\sum_{j=1}^{n} f_j(\mathbf{x}),$ problem $F3$ is obviously equivalent to $F2$. Therefore, we can solve $F3$ instead of $F2$. Thus Pareto maximization and a maximin problem are equivalent.■

Lemma 4.1.1 and the results of [6] establish the following result.

Theorem 4.1.2 Every SOC is criteria equivalent of every SOC.

4.2 Equivalent Scalarizations of SOCs

Let S_A and S_B be standard scalarizations of the given SOC A and SOC B respectively. It is said that both S_A is an equivalent scalarization of S_B and S_B is an equivalent scalarization of S_A if and only if they have same set of solutions. In other words, all solutions and only solutions of S_A can be obtained by solving S_B and vice versa.

Corollary 4.2.1 Every scalar equivalence of one SOC is an equivalent scalarization to a scalar equivalence of any other SOC.

Proof. Given any SOC problem, say SOC A. We denote the standard scalar equivalence of the SOC A as S_A. Consider any other SOC, say SOC B. By Theorem 4.1.2, SOC A is criteria equivalent to SOC B. Therefore, there is a problem involving SOC B that provides all solutions and only solutions to the given SOC A. It follows that all solutions and only solutions of the SOC B can be obtained by solving S_A. Hence, S_A and S_B have the same set of solutions. S_A is an equivalent scalarization to S_B and vice versa. Thus S_B is also another scalarization to the given SOC A.■

Example 4.2.2 Maximin Scalarization as Pareto Scalarization

Let the problem $G1$ below be the equivalent scalarization to a given maximin problem.

$$G1: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v \le f_1(\mathbf{x}) \\ & \vdots \\ & v \le f_n(\mathbf{x}) \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\} \text{ where } f_i : R^n \to R \text{ for } i = 1, \ldots, n.$$

We write $G1$ as the equivalent scalarization $G2$ below of Pareto maximization. For $i = 1,\ldots, n,$ let

$g_i(\mathbf{x}, v) = \dfrac{v}{\lambda_i}$ for all $\mathbf{x} \in A \subset R^n$ and $v \in R,$ where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1.$

Define $A_1 = \{(\mathbf{x}, v) \in R^{n+1} : (\mathbf{x}, v)$ is a feasible solution to $G1\}$, so the set A_1 is exactly the feasible region of $G1$. Now an equivalent scalarization for Pareto maximization of the n-objective function of $(g_1,\ldots, g_n)$ is given below as $G2$.

$$G2(y_1,\ldots,y_n): \left\{ \begin{array}{ll} \max\limits_{\mathbf{x},v} & \lambda_1 g_1(\mathbf{x},v) + \ldots + \lambda_n g_n(\mathbf{x},v) \\ \text{s.t.} & g_1(\mathbf{x},v) = \dfrac{v}{\lambda_1} \ge y_1 \\ & \vdots \\ & g_n(\mathbf{x},v) = \dfrac{v}{\lambda_n} \ge y_n \\ & (\mathbf{x},v) \in A_1 \end{array} \right\} \text{ for all } y_1,\ldots,y_n \in R.$$

Example 4.2.3 Pareto Scalarization as Maximin Scalarization

Let $f_i : R^m \to R$ for $i = 1,\ldots, n,$ where n is a positive integer. We write $H1$ below as the equivalent scalarization of [8] for Pareto maximization.

$$H1(y_1,\ldots,y_n): \left\{ \begin{array}{ll} \max\limits_{\mathbf{x}} & \lambda_1 f_1(\mathbf{x}) + \ldots + \lambda_n f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) \ge y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \ge y_n \\ & \mathbf{x} \in A_1 \subset R^m \end{array} \right\} \text{ for all } y_1,\ldots,y_n \in R.$$

Define $A_2(y_1,\ldots,y_n) = \{\mathbf{x} \in A_1 : \mathbf{x}$ is a feasible solution to $H1(y_1,\ldots,y_n)\}$ for $y_1,\ldots,y_n \in R$.
Obviously, the set $A_2(y_1,\ldots,y_n)$ is the set of feasible solutions of $H1(y_1,\ldots,y_n)$ for $y_1,\ldots,y_n \in R$. Consider the following n functions

$$g_1(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}),\ g_2(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}) + 2,\ \ldots,\ g_n(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}) + n,$$

$$\text{for all } \mathbf{x} \in A(y_1,\ldots,y_n).$$

Notice that $g_1(\mathbf{x}) < g_2(\mathbf{x}) < \ldots < g_{n-1}(\mathbf{x}) < g_n(\mathbf{x})$ for all $\mathbf{x} \in A(y_1,\ldots,y_n)$. Write the equivalent scalarization $F2(y_1,\ldots,y_n)$ below of the maximin of the $g_1(\mathbf{x}),\ldots,g_n(\mathbf{x})$.

$$H2(y_1,\ldots,y_n): \left\{ \begin{array}{lc} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v \le g_1(\mathbf{x}) \\ & \vdots \\ & v \le g_n(\mathbf{x}) \\ & \mathbf{x} \in A_2(y_1,\ldots,y_n), v \in R \end{array} \right\} \text{ for all } y_1,\ldots,y_n \in R$$

According to Sections 4.1 and 4.2 above, we summarize our results in Figure 4.1 below.

1. Given any problem involving SOC A, we can formulate a problem of SOC B to provide the same set of solutions. Therefore, all solutions and only solutions of the given SOC A can be obtained via an SOC B. On the other hand, all solutions and only solutions of a given problem of SOC B can be also obtained via an SOC A.
2. We can find all solutions and only solutions of the scalar equivalence of SOC A by solving a scalar equivalence of SOC B. On the contrary, we can solve the scalar equivalence of SOC B by solving the scalar equivalence of SOC A. Consequently, all solutions and only solutions of any given SOC can be obtained by solving a scalar equivalence of any type of SOC.

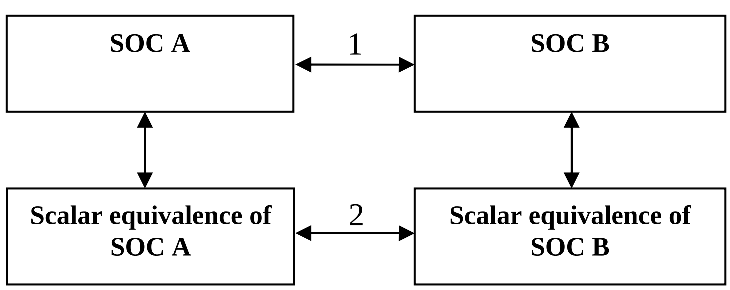

Figure 4.1 Equivalence Diagram

5. SOCs as Decision Problems

Since an optimization problem is a decision making problem for obtaining the "best" outcome according to a specific criterion, we consider here an SOC problem as a formal decision problem. In computability theory a decision problem [16] is a question in some formal logical system that can be answered by a yes-or-no answer. For example, the satisfiability problem SAT is a decision problem, whose instance is a Boolean expression using only AND, OR, NOT, variables, and parentheses. The question is then, given such a formula, is there some assignment of *TRUE* and *FALSE* values to the variables that will make the entire expression true? If the answer is "yes," the formula is satisfiable. If not, the formula is unsatisfiable. SAT is NP-complete [16] and thus may be solved as an 0-1 integer programming (IP) problem [24], which is simply a scalarization of SAT. All solutions of this 0-1 IP yield corresponding truth values of the Boolean variables that make the formula true, and only solutions of the IP correspond to truth values of the Boolean variables making it true. Another example of a formal decision problem is the halting problem. For a given computer program, the problem is to decide if the program will stop or continue running forever. It is well known that the halting problem is undecidable [25]; i.e., there is no an efficient algorithm that always provides a correct yes-or-no answer to the problem.

In a somewhat similar fashion, any SOC has a corresponding decision problem that asks whether there is a feasible solution yielding an objective function value of z_0 or better in the appropriate mathematical space. This question can be answered as "yes" or "no." Thus varying z_0 and solving a series of the associated decision problems provides either an arbitrarily close approximation to the answer of the SOC problem or gives an exact answer. To connect scalarization and formal decision problems at a more fundamental level, one may tentatively assert that scalarization is explicitly or implicitly a natural part of any decision-making process. As noted in [21] and [19], it is a standard human approach for making judgments by quantifying the options in a decision. Thus the converse question arises: can any formal decision problem be evaluated as a scalar maximization problem as in the case of SAT?

Let *D* be the set of decision problems, and let *S* be the set of decision problems that can be solved or deemed undecidable by the scalar maximization of a real-valued function over an appropriate feasible set. The question can then be restated as: does *D* = *S*? If *S* is a strict subset of *D*, one can ask whether a particular decision problem *d* in *D* is a member of *S*. Thus the fact that any SOC is equivalent and scalarizable leads to significant new questions in computability theory.

6. Conclusions and Future Work

For the SOCs maximin, Pareto optimization, goal programming, cone-ordered optimization, and set-valued optimization, scalar equivalences have been proposed here without convexity or concavity requirements. We have also shown that there is a direct equivalence between two SOCs. Thus all solutions and only solutions of any SOC problem can be obtained by solving a problem involving any other SOC. In addition, we have shown that a scalar equivalence of any SOC is similarly equivalent to any scalar equivalence for another SOC. Finally, we considered SOC problems as decision problems and asked the open question if any formal decision problem is scalarizable, i.e., can be solved by a real-valued maximization over an appropriate feasible region. Future research will unify the notion of an optimization criterion within a general axiomatic framework and seek new optimization criteria with meaningful applications.

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