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General Optimization Criteria

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Abstract

We define here the notion of a general optimization criterion (GOC). A set of axioms for GOC is proposed and discussed, and a scalar equivalence is presented. Examples of optimization criteria are then presented, as well as a decision rule that is not one. Finally, the new optimization criterion "compromise" is developed.

Keywords: Multiple criteria analysis, Optimization criteria, Scalarization

1. Introduction

Our previous paper [6] showed that any standard optimization criterion (SOC) is scalarizable. In other words, all solutions and only solutions of the problem can be achieved as the maximization (or equivalently a minimization) of a real-valued objective function subject to certain constraints. No further assumptions were required. In particular, Moreover, all SOCs are equivalent, and any problem involving one criterion can be formulated as a problem involving any other. This result provides the motivation to define a general optimization criterion (GOC), which includes all SOCs as special cases. Such work as [15] and [17], for example, is fundamentally different.

We present an axiomatic definition of a GOC to provide a consistent framework in which to describe a decision as being a "best" one. Next we develop a scalar equivalence for a GOC in the following sense. All solutions of the original problem and only solutions to it can be obtained via the maximization of a related real-valued function, i.e., a scalarization of the original problem. SOCs are shown to be examples of GOC. We then show that not all familiar decision-making problems involve a formal decision criterion with a counterexample in voting. Finally we construct a new optimization criterion called "compromise" for multi-objective optimization.

2. General Optimization Criteria

In this section we state our GOC axioms for preference relations in terms of partial orders, then give some examples and results.

2.1 Preference Orders

Consider a binary relation $\prec$ on R^n, i.e., $\prec \subset R^n \times R^n$, where it is not the case that $\mathbf{y} \prec \mathbf{y}, \forall \mathbf{y} \in R^n$. Next extend the strict order $\prec$ to $\preceq$ such that $\mathbf{y}_1 \preceq \mathbf{y}_2$ where $\mathbf{y}_1, \mathbf{y}_2 \in R^n$ if either $\mathbf{y}_1 \prec \mathbf{y}_2$ or $\mathbf{y}_1 = \mathbf{y}_2$. The order $\preceq$ is called a preference order [19]. In this definition, the strict relation $\mathbf{y}_1 \prec \mathbf{y}_2$ may not exist. However $\mathbf{y}_1 \preceq \mathbf{y}_2$ can be defined whenever $\mathbf{y}_1 = \mathbf{y}_2$. We say that $\mathbf{y}_2$ is preferred at least as much as $\mathbf{y}_1$ whenever $\mathbf{y}_1 \preceq \mathbf{y}_2$. If $\mathbf{y}_2$ is preferred more than $\mathbf{y}_1$, $\mathbf{y}_1 \prec \mathbf{y}_2$.

2.2 General Optimization Problems

Given a preference order $\preceq$ on R^n and $\mathbf{y}_1, \mathbf{y}_2 \in R^n$, $\mathbf{y}_2$ dominates $\mathbf{y}_1$ if $\mathbf{y}_1 \preceq \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. A vector $\mathbf{y}_1 \in A \subset R^n$ is said to be non-dominated in A if there is no $\mathbf{y}_2 \in A$ such that $\mathbf{y}_1 \preceq \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. Denote the set $\text{opt}\, A$ as the set containing all non-dominated vectors in A with respect to the preference order $\preceq$.

For a preference order $\preceq$ on R^n, consider the general optimization problem

$$\text{(G)} \quad \left\{ \begin{array}{l} \text{opt}_{\mathbf{x}} \; f(\mathbf{x}) \\ \text{s.t. } \mathbf{x} \in A \subset R^m \end{array} \right\},$$

with objective function $f : R^m \to R^n$ and feasible region A. We seek a vector $\mathbf{x}^* \in A \subset R^m$ for which there is no vector $\mathbf{x} \in A$ such that $f(\mathbf{x}^*) \prec f(\mathbf{x})$, or equivalently that $f(\mathbf{x}^*) \preceq f(\mathbf{x})$ and $f(\mathbf{x}^*) \neq f(\mathbf{x})$. Such an $\mathbf{x}^* \in A \subset R^m$ is called an optimal solution to the problem. Denote $\text{opt}\; f(A)$ as the set of all optimal objective values, which could be empty.

Example 2.2.1 The Pareto maximization [12], [13], and [18], $\operatorname{Vmax}_{\mathbf{x}\in A} f(\mathbf{x})$, where $f: R^m \to R^n$ and $A \subset R^m$, is a special case of the general optimization problem where the preference order $\leq_{Pareto}$ in R^n is defined as following: $(a_1,...,a_n) \leq_{Pareto} (b_1,...,b_n)$ if and only if $a_i \leq b_i$ for all $i = 1,\ldots,n$ and $a_j < b_j$ for some index j where $a_i, b_i \in R$ for all $i = 1,\ldots,n$.

2.3 Axioms for General Optimization Criteria

Given any optimization problem G in R^n, "opt" is a formal $\preceq$ optimization criterion on R^n if the following two axioms are satisfied.

Axiom 1: Axiom of Partial Order (APO)

The preference order $\preceq$ in the objective space $f(A) \subset R^n$ is a partial order [3]. Equivalently, $\preceq$ satisfies the following three properties.

1. **Reflexive** property: $\mathbf{x} \preceq \mathbf{x}$ for all $\mathbf{x} \in f(A)$.
2. **Antisymmetric** property: If $\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{x}$ for any $\mathbf{x}$, $\mathbf{y} \in f(A)$, then $\mathbf{x} = \mathbf{y}$.
3. **Transitive** property: If $\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{z}$ for any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in f(A)$, then $\mathbf{x} \preceq \mathbf{z}$.

Axiom 2: Axiom of Scalarizability Property (ASP)

G has a scalar equivalence. In other words, all solutions and only solutions of the optimization problem can be obtained by solving a real-valued maximization problem subject to appropriate constraints.

2.4. Discussion of Axioms

The goal of Axioms 1 and 2 is to provide a consistent decision-making framework that yields identical optimal decisions in identical situations for a large class of applications. In practice, people may make preference decisions using methods not satisfying our axioms, such as the voting scheme of Section 4.5. However, such methods are not considered as optimization criteria according to our definition.

2.4.1 Axiom of Partial Order

We first note that APO alone provides meaningful generalizations in [5] of the following two properties of real-valued maximizations.

1. **Inclusion property**: Let A and B be subsets of R^m such that $A \subset B$. Then we have

   $\max_{\mathbf{x}\in A} f(\mathbf{x}) \le \max_{\mathbf{x}\in B} f(\mathbf{x})$ where $f: R^m \to R.$

2. **Triangle inequality for maximization**:

   $\max_{\mathbf{x}\in A\subset R^m} (f+g)(\mathbf{x}) \le \max_{\mathbf{x}\in A\subset R^m} f(\mathbf{x}) + \max_{\mathbf{x}\in A\subset R^m} g(\mathbf{x}),$ where $f, g: R^m \to R^n.$

Next, since no decision choice $\mathbf{x}$ should be preferred more or less than itself (i.e., $\mathbf{x} \prec \mathbf{x}$), the preference order for a decision must have the reflexive property of a partial order. Finally, the difficulty of reasonable a reasonable choice without the nantisymmetric and transitive properties is illustrated in the following two examples.

Example 2.4.1.1 Consider the relation order $\preceq$ on the set {3, 5} such that 3=3, 5=5, $3 \prec 5$ and $5 \prec 3.$ This order lacks the antisymmetric property because 3 does not identically equal 5. Also, it is contradictory for the decision maker to simultaneously evaluate $3 \prec 5$ and $5 \prec 3.$ In addition, there is no "best" value or values to choose, though each value is compared to each value. Hence, the antisymmetric property seems a reasonable requirement.

Example 2.4.1.2 Consider the relation order $\preceq$ on the set of {5, 8, 10} such that 5=5, 8=8, 10=10, $5 \prec 8$, $8 \prec 10$, and $10 \prec 5$ and is thus not transitive. Again, there is no best value or values to choose. In this case, the reason is that 8 is "better" than 5, 10 is "better" than 8, but 5 is "better" than 10. A decision maker using such a preference order would be inconsistent.

Such intransitivity can occur in elections. A voter may prefer candidate A to B, B to C, and C to A. The difficulty is that if a selection were conducted by successive pairwise comparison, then a different "best" candidate would be chosen for different pairwise comparisons. it seems more reasonable that the simultaneous comparisons of candidates should give the same result as sequential pairwise comparisons in a decision framework that purports to select a "best" solution. The transitivity dictated by APO avoids such inconsistencies. Of course, decisions can be made without this property, but the term "optimal" cannot be applied within our framework.

2.4.2 Axiom of Scalarizability Property (ASP)

One reason that ASP seems reasonable is that one can always define a utility function on a set of choices [19]. Furthermore, all standard optimization criteria (SOC) are scalarizable [5], so ASP is a natural extension. The determining reason, though, was that we were unable to construct a problem G with respect to a partial order in R^n for which it could be proved that no scalar equivalence exists for finding all solutions and only solutions for G. The existence of such a counterexample remains an open question. One candidate is presented in Example 2.4.2.1, for which we could obtain all and only solutions of G by a family of scalarizations, not a single real-valued maximization problem as required by ASP.

Example 2.4.2.1 Define a be a partial order $\preceq$ on R^2 as follows. For each $\mathbf{x} \in R^2$, let $I(\mathbf{x}) = \{\mathbf{y} \in R^2 : \mathbf{x} \preceq \mathbf{y}\}$ be the set having **x** as the first element. Construct the collection $C(\mathbf{x})$ containing all chains in $I(\mathbf{x})$ with **x** as the first element by setting $C(\mathbf{x}) = \{ P_i(\mathbf{x}) \subset I(\mathbf{x}) : i \in \wedge_{\mathbf{x}} \}$, where $\wedge_{\mathbf{x}}$ is an index set and $P_i(\mathbf{x})$ has the following properties:

1. $\forall \mathbf{y}_1, \mathbf{y}_2 \in P_i(\mathbf{x}), \mathbf{y}_1 \preceq \mathbf{y}_2$ or $\mathbf{y}_2 \preceq \mathbf{y}_1$,
2. $\mathbf{x} \in P_i(\mathbf{x})$.

According to Lemma 2.4.2.2 below, we have $(R^2, \preceq) = (\bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x})}} P_i(\mathbf{x}), \preceq)$. In other words, $(R^2, \preceq)$ can be decomposed into an uncountable union of chains.

Lemma 2.4.2.2 $(R^2, \preceq) = (\bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x}) \\ i \in \wedge_{\mathbf{x}}}} P_i(\mathbf{x}), \preceq)$.

Proof. By the above construction, $P_i(\mathbf{x}) \subset I(\mathbf{x}) \subset R^2$ for all $\mathbf{x} \in R^2$, $i \in \wedge_{\mathbf{x}}$, and so $(\bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x}) \\ i \in \wedge_{\mathbf{x}}}} P_i(\mathbf{x}), \preceq) \subset (R^2, \preceq)$. To prove the opposite inclusion, let $\mathbf{y} \in R^2$. Since $\mathbf{y} \in P_i(\mathbf{y})$ for all $i \in \wedge_{\mathbf{y}}$, we have that $\mathbf{y} \in P_i(\mathbf{y}) \subset \bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x}) \\ i \in \wedge_{\mathbf{x}}}} P_i(\mathbf{x})$, so $R^2 \subset \bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x})}} P_i(\mathbf{x})$. Now let $(\mathbf{x}, \mathbf{y}) \in (R^2, \leq)$, i.e., $\mathbf{x} \preceq \mathbf{y}$. Then $(\mathbf{x}, \mathbf{y}) \in (P_i(\mathbf{x}), \preceq)$ for some $i \in \wedge_{\mathbf{x}}$ by definition. The conclusion now follows that

$$(R^2, \preceq) \subset (\bigcup_{\substack{\mathbf{x} \in R^2 \\ P_i(\mathbf{x}) \in C(\mathbf{x}) \\ i \in \wedge_{\mathbf{x}}}} P_i(\mathbf{x}), \preceq). \blacksquare$$

Consider the following general optimization $A1$, for which "opt" may not represent an axiomatically formal optimization criterion.

$$A1: \left\{ \begin{array}{l} \operatorname{opt}_{\mathbf{x}} \ f(\mathbf{x}) \\ \text{s.t.}\ \mathbf{x} \in A \subset R^m \end{array} \right\}, \text{ where } f: R^m \to R^2, \text{ and } R^2 \text{ has a partial order } \preceq.$$

Construct the uncountable family $A2$ of scalar maximizations parameterized by $\mathbf{w} \in R^2$ as

$$A2: \left\{ \begin{array}{ll} \max_{\mathbf{x}} & l_{\mathbf{w}}^i(f(\mathbf{x})) \\ \text{s.t.} & f(\mathbf{x}) \in P_i(\mathbf{w}) \\ & \mathbf{x} \in A \end{array} \right\},$$ where $\mathbf{w} \in R^2, i \in \wedge_{\mathbf{w}}$ and $l_{\mathbf{w}}^i$ is a real-valued function mapping from R^2 with the following property. If $f(\mathbf{x}) \prec f(\mathbf{y})$ for $f(\mathbf{x}), f(\mathbf{y}) \in P_i(\mathbf{w})$ for each $\mathbf{w} \in R^2, i \in \wedge_{\mathbf{w}}$, then $l_{\mathbf{w}}^i(f(\mathbf{x})) < l_{\mathbf{w}}^i(f(\mathbf{y}))$.

Theorem 2.4.2.3 If $\mathbf{x}_0$ solves $A1$ then $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and all $i \in \wedge_{\mathbf{w}}$.

Proof. Assume that $\mathbf{x}_0$ solves $A1$. Since $\mathbf{w} = f(\mathbf{x}_0)$, $\mathbf{x}_0$ is a feasible solution to $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and any $i \in \wedge_{\mathbf{w}}$. Let $\mathbf{x}_1$ be any feasible solution to $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and $i \in \wedge_{\mathbf{w}}$. Since $\mathbf{x}_0$ solves $A1$, it must be that $f(\mathbf{x}_1) = f(\mathbf{x}_0)$. Thus every feasible point of $A2(f(\mathbf{x}_0), i)$ is a solution as well. But $\mathbf{x}_0$ is a feasible to $A2(f(\mathbf{x}_0), i)$, so it solves $A2(f(\mathbf{x}_0), i)$. $\blacksquare$

Theorem 2.4.2.4 If $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} \in R^n$ and any $i \in \wedge_{\mathbf{w}}$, then $\mathbf{x}_0$ solves $A1$.

Proof. Assume that $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} \in R^n$ and $i \in \wedge_{\mathbf{w}}$. To obtain a contradiction, suppose that $f(\mathbf{x}_0) \notin \operatorname{opt} f(A)$. Then there exists $\mathbf{x}_1 \in A$ such that $f(\mathbf{x}_0) \prec f(\mathbf{x}_1)$. Otherwise $f(\mathbf{x}_0) \in \operatorname{opt} f(A)$. Since $f(\mathbf{x}_0) \in P_i(\mathbf{w})$, it follows that $f(\mathbf{x}_1) \in P_i(\mathbf{w})$ by the definition of $P_i(\mathbf{w})$. Hence $\mathbf{x}_1$ is feasible to $A2$. But $f(\mathbf{x}_0) \prec f(\mathbf{x}_1)$, so $l_{\mathbf{w}}^i(f(\mathbf{x}_0)) < l_{\mathbf{w}}^i(f(\mathbf{x}_1))$ in contradiction to the optimality of $\mathbf{x}_0$. $\blacksquare$

Definition 2.4.2.5 [4]. Let $(R^n, \trianglelefteq)$ be a partially ordered set. We say that the preorder $\trianglelefteq$ is order separable in the sense of Cantor if there exists a countable subset $Z \subset R^n$ such that whenever $\mathbf{x} \prec \mathbf{y},$ there exists $\mathbf{z} \in Z$ such that $\mathbf{y} \prec \mathbf{z} \prec \mathbf{x}.$ In particular, preorders [3] are binary relations that are reflexive and transitive.

Theorem 2.4.2.6 [4]. Let $(R^n, \trianglelefteq)$ be a partially ordered set that is order separable in the sense of Cantor. Then there is a real-valued function *f* on R^n such that $\mathbf{y}_1 \prec \mathbf{y}_2$ implies $f(\mathbf{y}_1) < f(\mathbf{y}_2).$ Such a real-valued function *f* is called a strictly monotone functional on $(R^n, \trianglelefteq).$

Under the existence of $l_{\mathbf{w}}^i$ for all $i \in \wedge_{\mathbf{w}}$ and $\mathbf{w} \in R^2,$ problems $A2$ and $A1$are equivalent as a consequence of Theorems 2.4.2.3 and 2.4.2.4. All solutions and only solutions of $A1$ can be theoretically obtained by $A2$ and vice versa. According to Theorem 2.4.2.6, separability in the sense of Cantor of all chains $P_i(\mathbf{w})$ in R^2 guarantees the existence of a strictly monotone function $l_{\mathbf{w}}^i.$ However, the objective function $l_{\mathbf{w}}^i(f(\mathbf{x}))$ may obviously be different from $l_{\mathbf{w}}^j(f(\mathbf{x}))$ where $i, j \in \wedge_{\mathbf{w}}$, or different from $l_{\mathbf{y}}^i(f(\mathbf{x}))$ where $i \in \wedge_{\mathbf{y}}$ for $\mathbf{w}, \mathbf{y}, \mathbf{z} \in R^2.$ Therefore $A2$ is not considered as an equivalent scalarization of $A1$ since there is no common objective function for the family.

3. A Scalarization for a General Optimization Criterion (SGOC)

We now present a scalar equivalence for any GOC problem satisfying Assumption 3.1.1 below. In other words, all solutions and only solutions to a general optimization problem involving the original criterion can be obtained by certain scalar maximization problems and vice versa. Any such scalar equivalence is required to be a real-valued maximization subject to either (i) a fixed feasible region or (ii) a parameterized feasible region for which maximization are to be performed for all parameters in a given parameter set . In both (i) and (ii), the feasible region is usually determined by a set of constraints.

The SGOC of this paper is motivated by the scalarization for the lexicographic optimization criterion given in [7]. Here we define a strictly monotone real-valued function corresponding to each of *n* components when the others are fixed. Using the fact that only partial orders $\trianglelefteq$ in R^n are considered, we initially construct *n* induced orders corresponding, respectively, to the *n* component of R^n. We then utilize Theorem 2.4.2.6 to provide a strictly monotone functional corresponding to each component in R^n with the other components held fixed.

3.1 Component Orders

Consider a partial order $\preceqq$ in R^n. For each $1 \le m \le n,$ define an induced order $\preceqq^m$ on R corresponding to the m^{th} component of vectors in R^n as follows. Denote

$$a_m \preceqq^m b_m \text{ if and only if } (0,...,a_m,...,0) \preceqq (0,...,b_m,...,0) \text{ for } a_m, b_m \in R.$$

We first show that the induced order $\preceqq^m$ is partially ordered.

Theorem 3.1.1 The induced order $\preceqq^m$ is a partial order in R for any $1 \le m \le n.$

Proof. Let $m \in \{1,..., n\}$. We show that $\preceqq^m$ is reflexive, antisymmetric, and transitive.

(Reflexive). Let $a_m \in R.$ Since $\preceqq$ is a reflexive in R^n, $(0,...,a_m,...,0) \preceqq (0,...,a_m,...,0).$ Then by definition $a_m \preceqq^m a_m.$

(Antisymmetric). Let $a_m, b_m \in R$ such that $a_m \preceqq^m b_m$ and $b_m \preceqq^m a_m.$ By definition, we have that $(0,...,a_m,...,0) \preceqq (0,...,b_m,...,0)$ and $(0,...,b_m,...,0) \preceqq (0,...,a_m,...,0).$ Since $\preceqq$ is antisymmetric, $(0,...,a_m,...,0) = (0,...,b_m,...,0).$ It follows that $a_m = b_m.$

(Transitive). Let $a_m, b_m, c_m \in R$ such that $a_m \preceqq^m b_m$ and $b_m \preceqq^m c_m.$ By definition we also have $(0,..., a_m,...,0) \preceqq (0,...,b_m,...,0)$ and $(0,...,b_m,...,0) \preceqq (0,...,c_m,...,0).$ Since $\preceqq$ is transitive, $(0,...,a_m,...,0) \preceqq (0,...,c_m,...,0)$ and thus $a_m \preceqq^m c_m.$

It now follows that $\preceqq^m$ is a partial order in R for $1 \le m \le n$. ■

Since $(R, \preceqq^m)$ is separable in the sense of Cantor, whereas R^n is not, Theorem 2.4.2.6 provides an immediate corollary Theorem 3.1.1.

Corollary 3.1.2. There exists a strictly monotone function $l^m : R \to R$ with respect to the order $\preceqq^m$ for any $1 \le m \le n$.

3.2 SGOC Formulation

Our SGOC requires the partial order $\preceqq$ of APO to satisfy Assumption 3.2.1 below.

Assumption 3.2.1. Let $\preceqq$ be a partial order on R^n and $\preceqq^m$ be its induced component order in Section 3.1 for all $1 \le m \le n$. We assume that the following statements are true.

If $(a_1, a_2, ..., a_n) \preceqq (b_1, b_2, ..., b_n)$, then $a_1 \preceqq^1 b_1$.

If $(c_1, a_2, ..., a_n) \preceqq (c_1, b_2, ..., b_n)$ for some $c_1 \in R$, then $a_2 \preceqq^2 b_2$.

If $(c_1, c_2, a_3, ..., a_n) \preceqq (c_1, c_2, b_3, ..., b_n)$ for some $c_1, c_2 \in R$, then $a_3 \preceqq^3 b_3$.

$$\vdots$$

If $(c_1, c_2, c_3, ..., c_{n-1}, a_n) \preceqq (c_1, c_2, c_3, ..., c_{n-1}, b_n)$ for some $c_1, c_2, ..., c_{n-1} \in R$, then $a_n \preceqq^n b_n$.

It should be noted that the usual orders induced by the Pareto and lexicographic cones [1] satisfy Assumption 3.2.1. Moreover, SGOC will generalize the equivalent scalarizations of [6] for Pareto and lexicographic maximization.

Let $f_i : R^m \to R, i = 1, ..., n, A \subset R^m$, and consider the GCOP $B1$: $\underset{x \in A}{\text{opt}}(f_1(\mathbf{x}), ..., f_n(\mathbf{x}))$, whose associated partial order $\preceqq$ satisfies Assumption 3.2.1. Then an equivalent SGOC scalarization problem for $B1$ is given by

$$B2(\mathbf{y}): \left\{ \begin{array}{ll} \max\limits_{\mathbf{x}} & l^n(f_n(\mathbf{x})) = (0, ..., 0, 1)^T \cdot (l^1(f_1(\mathbf{x})), ..., l^n(f_n(\mathbf{x}))) \\ \text{s.t.} & f(\mathbf{x}) \succeqq \mathbf{y} \\ & l^1(f_1(\mathbf{x})) = a_1(\mathbf{y}) \\ & \vdots \\ & l^{n-1}(f_{n-1}(\mathbf{x})) = a_{n-1}(\mathbf{y}) \\ & \mathbf{x} \in A \end{array} \right\} \text{ for all } \mathbf{y} \in f(A),$$

where $a_m(\mathbf{y}), m = 1, ..., n-1,$ are the optimal objective function values of the following problems.

$$B(\mathbf{y},1):\left\{\begin{array}{ll}\max & l^1(f_1(\mathbf{x}))\\ \text{s.t.} & f(\mathbf{x})\succeqq\mathbf{y}\\ & \mathbf{x}\in A\end{array}\right\} \text{ for } m=1, \text{ and}$$

$$B(\mathbf{y},m):\left\{\begin{array}{ll}\max\limits_{\mathbf{x}} & l^m(f_m(\mathbf{x}))\\ \text{s.t.} & l^1(f_1(\mathbf{x}))=a_1(\mathbf{y})\\ & \\ & l^{m-1}(f_{m-1}(\mathbf{x}))=a_{m-1}(\mathbf{y})\\ & f(\mathbf{x})\succeqq\mathbf{y}\\ & \mathbf{x}\in A\end{array}\right\} \text{ for } 2\le m\le n-1.$$

Lemma 3.2.2 If $\mathbf{x}_0$ solves $B1$, then $\mathbf{x}_0$ is feasible to $B2(\mathbf{y})$ for $\mathbf{y}=f(\mathbf{x}_0)$.

Proof. Let $\mathbf{x}_0$ solve $B1$. By the optimality of $\mathbf{x}_0$, whenever $f(\mathbf{x})\succeqq f(\mathbf{x}_0)$ for $\mathbf{x}\in A$, then $f(\mathbf{x})=f(\mathbf{x}_0)$. Thus $\mathbf{x}_0$ solves $B(\mathbf{y},1),\ldots,B(\mathbf{y},n-1)$, where $\mathbf{y}=f(\mathbf{x}_0)$. It follows that $l^k(f_k(\mathbf{x}_0))=a_k(\mathbf{y}=f(\mathbf{x}_0))$ for any $1\le k\le n-1$. Hence, $f(\mathbf{x}_0)\succeqq f(\mathbf{x}_0)$, so $\mathbf{x}_0$ is feasible to $B2(\mathbf{y})$ for $\mathbf{y}=f(\mathbf{x}_0)$. ■

Theorem 3.2.3 If $\mathbf{x}_0$ solves $B1$, then $\mathbf{x}_0$ solves $B2(\mathbf{y})$ for $\mathbf{y}=f(\mathbf{x}_0)$.

Proof. Assume $\mathbf{x}_0$ solves $B1$. By Lemma 3.2.3, $\mathbf{x}_0$ is feasible to $B2(f(\mathbf{x}_0))$; i.e., $l^1(f_1(\mathbf{x}_0))=a_1,\ldots,l^{n-1}(f_{n-1}(\mathbf{x}_0))=a_{n-1}$, and $f(\mathbf{x}_0)\succeqq f(\mathbf{x}_0)$. To obtain a contradiction, suppose $\mathbf{x}_0$ does not solve $B2\,f(\mathbf{x}_0)$. Then there exists a feasible solution $\mathbf{x}_1\in A$ such that $l^n(f_n(\mathbf{x}_1))>l^n(f_n(\mathbf{x}_0))$. Since $\mathbf{x}_1$ is feasible to $B2\,f(\mathbf{x}_0)$, then $f(\mathbf{x}_1)\succeqq f(\mathbf{x}_0)$ and $f(\mathbf{x}_1)\ne f(\mathbf{x}_0)$. It follows that $f(\mathbf{x}_1)\succ f(\mathbf{x}_0)$, in contradiction to the optimality of $\mathbf{x}_0$. ■

Theorem 3.2.4 If $\mathbf{x}_0$ solves $B2(\mathbf{y})$ for $\mathbf{y}\in f(A)$, then $\mathbf{x}_0$ solves $B1$.

Proof. Suppose $\mathbf{x}_0$ solve $B2(\mathbf{y})$ for $\mathbf{y}\in f(A)$. Then $\mathbf{x}_0$ is feasible to $B1$ and $\mathbf{y}\preceqq(f_1(\mathbf{x}_0),\ldots,f_n(\mathbf{x}_0))$. Now let $\mathbf{x}_1$ be any feasible solution to $B1$ such that $\mathbf{y}\preceqq(f_1(\mathbf{x}_0),\ldots,f_n(\mathbf{x}_0))\preceqq(f_1(\mathbf{x}_1),\ldots,f_n(\mathbf{x}_1))$. From Assumption 3.2.1, $f_1(\mathbf{x}_0)\preceqq^1 f_1(\mathbf{x}_1)$. But $l^1(f_1(\mathbf{x}_0))=\max\{l^1(f_1(\mathbf{x})): f(\mathbf{x})\succeqq\mathbf{y},\mathbf{x}\in A\}$, so

$f_1(\mathbf{x}_0) = f_1(\mathbf{x}_1)$. Then again by Assumption 3.2.1, we get that $f_2(\mathbf{x}_0) \leqq^2 f_2(\mathbf{x}_1)$ from which

$l^2(f_2(\mathbf{x}_0)) = \max\{l^2(f_2(\mathbf{x})) : f(\mathbf{x}) \geqq \mathbf{y}, l^1(f(\mathbf{x})) = l^1(f(\mathbf{x}_0)), \mathbf{x} \in A\}$ immediately yields

$f_2(\mathbf{x}_0) = f_2(\mathbf{x}_1)$. By applying a similar argument sequentially, we obtain that

$f_3(\mathbf{x}_0) = f_3(\mathbf{x}_1), ..., f_n(\mathbf{x}_0) = f_n(\mathbf{x}_1)$, respectively, so $\mathbf{x}_0$ solves $B1$.■

Example 3.2.5

Consider the following Pareto maximization problem

$\operatorname{Vmax}_{x \in A}(x_1^2, x_2^2)$ s.t. $A = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1, 0 \le x_1, x_2 \le 1\} \subset R^2$.

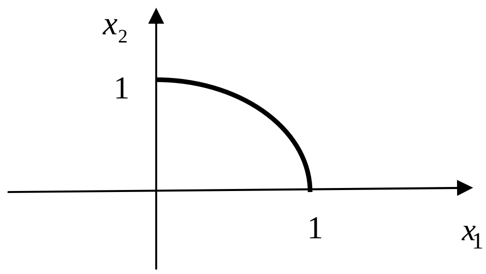

Figure 4.12 Pareto frontier of Example 3.2.6

We solve this Pareto problem with SGOC as follows. Define the induced orders on each component by $x_1 \le^1 y_1$ if and only if $(x_1, 0) \le_{Pareto} (y_1, 0)$ for $x_1, y_1 \in R,$ and $x_2 \le^2 y_2$ if and only if $(0, x_2) \le_{Pareto} (0, y_2)$ for $x_2, y_2 \in R.$ Formulate SGOC as

$$C1(y_1, y_2): \left\{ \begin{array}{c} \max_{x_1, x_2} \quad (0,1)^T \cdot (x_1, x_2) = (x_2) \\ x_1 \ge y_1 \\ x_2 \ge y_2 \\ x_1 = a_1(y_1, y_2) \\ x_1^2 + x_2^2 \le 1 \\ 0 \le x_1, x_2 \le 1 \end{array} \right\} \text{ for all } (y_1, y_2) \in f(A),$$

where $f(A) = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1, 0 \le x_1, x_2 \le 1\}$ and $a_1(y_1, y_2)$ is the optimal objective function value of the problem

$$C2(y_1, y_2): \left\{ \begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 \ge y_1 \\ & x_2 \ge y_2 \\ & x_1^2 + x_2^2 \le 1 \\ & 0 \le x_1, x_2 \le 1 \end{array} \right\}.$$

Select $(y_1, y_2) = (0.5, 0.5)$. By solving $C2(0.5, 0.5)$, we obtain $a_1(0.5, 0.5) = 0.866$. We then solve $C1(0.5, 0.5)$ and obtain the optimal solution $(x_1*, x_2*) = (0.866, 0.5)$. Notice that $(0.866, 0.5)$ is on the Pareto frontier in Figure 4.12 above. To obtain all solutions, we would need to solve $C1(y_1, y_2)$ for all values of y_1 and y_2. However, a representative sample would approximate the Pareto frontier.

4. Examples and Counterexample of GOC

We now verify any SOC satisfies the requirements for a GOC. SOCs include standard cone-ordered optimization [13], standard set-valued optimization [8], goal programming [7] and [16], and maximin optimization [9]. In particular, lexicographic optimization, Pareto optimization, and scalar optimization are special cases of cone-ordered optimization. Finally, an example of a decision rule that does not satisfy our axioms is presented as a counterexample.

4.1 Standard Cone-Ordered Maximization

The order in any standard cone-ordered maximization is a partial order because it is induced by a pointed convex cone [6]. Thus Axiom 1 (APO) is satisfied. In addition, if any standard cone-ordered optimization is scalarizable, then Axiom 2 (ASP) is satisfied. From results in [6], [8], and [9], if the pointed convex cone is also closed as in the Pareto case, then a cone-ordered optimization is scalarizable. The only standard cone-ordered maximization for the which the cone is not closed is the pointed convex lexicographic cone. However, it is also scalarizable as shown in [7]. Thus, any standard cone-ordered maximization represents a GOC.

4.2 Set-Valued Maximization

Consider the following standard set-valued maximization problem $\max_{\mathbf{x} \in A} F(\mathbf{x}),$ where $F: R^m \to 2^{R^n}$ is a point-to-set map and the order in R^n is induced by a closed pointed convex cone C in R^n. As before, APO and ASP are satisfied, so set-valued maximization is a GOC.

4.3 Maximin A maximin optimization is a scalar maximization where the objective function is simply a minimization itself [6]. Thus it is a GOC.

4.4 Goal Programming

Goal programming can be defined as a Pareto maximization [7], which is a GOC as above.

4.5 A Voting Counterexample

Consider the well-known Condorcet Paradox [2] and [20] in voting.

Table 4.1 Condorcet paradox.

Individual	Preference order
Voter 1	A>B>C
Voter 2	B>C>A
Voter 3	C>A>B

In this example, three voters, 1, 2, and 3 are asked to consider three alternatives A, B, and C. As shown in Table 4.1, Voter 1 prefers A to B to C; Voter 2 prefers B to C to A; and Voter 3 prefer C to A to B. It is obvious that two people prefer A to B, two people prefer B to C, and two people prefer C to A. A majority voting scheme immediately gives A < B and B < C, but C < A. This group preference order is thus intransitive and cannot be a partial order, so marity voting is not a formal optimization criterion.

5. A New Optimization Criterion

5.1 The Compromise Criterion

We now define a new optimization criterion on R^n that attempts to yield a compromise solution. This solution will also be shown to be a Pareto maximum.

Let $f : R^m \to R^n$ be a nonnegative objective function for which represents *n* outcomes. Assume that $-\infty < \min_{\mathbf{x}\in A} f_i(\mathbf{x})$ and $\max_{\mathbf{x}\in A} f_i(\mathbf{x}) < \infty$ for all *i*. Denote $M_i = \max_{\mathbf{x}\in A} f_i(\mathbf{x})$ and $m_i = \min_{\mathbf{x}\in A} f_i(\mathbf{x})$. Now define $T_{Compr} : f(A) \to R$ by

$$T_{Compr}(f(\mathbf{x})) = \left[\left(\frac{f_1(\mathbf{x}) - m_1 + 1}{M_1 - m_1 + 1}\right) \times \ldots \times \left(\frac{f_n(\mathbf{x}) - m_n + 1}{M_n - m_n + 1}\right)\right], \text{ for all } \mathbf{x} \in A.$$

Define a strictly compromise order on $f(A)$ as follows.

For any $f(\mathbf{x}_1), f(\mathbf{x}_2) \in f(A),$ $f(\mathbf{x}_1) <_{Compr} f(\mathbf{x}_2)$ if and only if $T_{Compr}(f(\mathbf{x}_1)) < T_{Compr}(f(\mathbf{x}_2)).$

Next, define the compromise order $\leq_{Compr}$ by

$$f(\mathbf{x}_1) \leq_{Compr} f(\mathbf{x}_2) \text{ if and only if } f(\mathbf{x}_1) <_{Compr} f(\mathbf{x}_2) \text{ or } f(\mathbf{x}_1) = f(\mathbf{x}_2).$$

A compromising problem can be written as $\underset{\mathbf{x}\in A}{\text{Compromise}}\ f(\mathbf{x})$ or $\underset{\mathbf{x}\in A}{\text{Opt}}\ f(\mathbf{x})$ with respect to $\leq_{Compr}$.

The problem is to find a vector $\mathbf{x}^* \in A \subset X$ for which there is no vector $\mathbf{x} \in A$ such that $f(\mathbf{x}^*) <_{Compr} f(\mathbf{x}),$ or equivalently that $f(\mathbf{x}^*) \leq_{Compr} f(\mathbf{x})$ and $f(\mathbf{x}^*) \neq f(\mathbf{x}).$

Lemma 5.1.1 For any $f(\mathbf{x}), f(\mathbf{y}) \in f(A),$ if $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$ then $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y})).$

Proof. Let $f(\mathbf{x}), f(\mathbf{y}) \in f(A),$ such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y}).$ Then, $0 \leq f_i(\mathbf{x}) \leq f_i(\mathbf{y})$ for all $i = 1,\dots,n$ and $0 \leq f_j(\mathbf{x}) < f_j(\mathbf{y})$ for some index *j*. Since all elements in $f(A)$ are nonnegative and definition of m_i and M_i , we have $0 \leq \dfrac{f_i(\mathbf{x}) - m_i + 1}{M_i - m_i + 1} \leq \dfrac{f_i(\mathbf{y}) - m_i + 1}{M_i - m_i + 1}$, for all $i = 1,\dots,n$, and $0 \leq \dfrac{f_j(\mathbf{x}) - m_j + 1}{M_j - m_j + 1} < \dfrac{b_j - m_j + 1}{M_j - m_j + 1}$ for some index *j*. It follows that

$$T_{Compr}(f(\mathbf{x})) = \prod_{i=1}^{n} \frac{f_i(\mathbf{x}) - m_i + 1}{M_i - m_i + 1} < \prod_{i=1}^{n} \frac{f_i(\mathbf{y}) - m_i + 1}{M_i - m_i + 1} = T_{Compr}(f(\mathbf{y})).$$
■

The next lemma shows that a compromise solution is also a Pareto maximum.

Lemma 5.1.2 If $f(\mathbf{x}) \in \text{Compromise } f(A),$ then $f(\mathbf{x}) \in \text{Vmax } f(A).$

Proof. Let $f(\mathbf{x}) \in \text{Compromise } f(A).$ To obtain a contradiction, suppose that $f(\mathbf{x}) \notin \text{Vmax } f(A).$ Then there exist $f(\mathbf{y}) \in f(A)$ such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y}).$ By Lemma 5.1.1, it follows that $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$ which contradicts with optimality of $f(\mathbf{x}).$ We conclude that $f(\mathbf{x}) \in \text{Vmax } f(A).$ ■

Lemma 5.1.3 $\text{Compromise } f(A) \subset \text{Vmax } f(A).$

Proof. It follows directly from Lemma 5.1.2. ■

Theorem 5.1.4 The preference order $\leq_{Compr}$ is a partial order on $f(A).$

Proof. We show that $\leq_{Compr}$ is reflexive, transitive, and antisymmetric.

(Reflexive). Since $f(\mathbf{x}) = f(\mathbf{x})$, we have $f(\mathbf{x}) \leq_{Compr} f(\mathbf{x})$ for any $f(\mathbf{x}) \in f(A)$.

(Transitive). Let $f(\mathbf{x}) \leq_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Compr} f(\mathbf{z})$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$.

Case 1: $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Pareto} f(\mathbf{z})$.

Since Pareto order is transitive, we have that $f(\mathbf{x})$ comparable to $f(\mathbf{z})$ and in particular $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Therefore, $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

Case 2: $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y})$ are not Pareto comparable with $f(\mathbf{z})$ with $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$.

Case 2.1: $f(\mathbf{x})$ is Pareto comparable with $f(\mathbf{z})$.

We claim that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Suppose that $f(\mathbf{z}) <_{Pareto} f(\mathbf{x})$. By Lemma 5.1.1., we have $T_{Compr}(f(\mathbf{z})) < T_{Compr}(f(\mathbf{x}))$. Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and by Lemma 5.1.1, we have $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. Therefore we obtain $T_{Compr}(f(\mathbf{z})) < T_{Compr}(f(\mathbf{y}))$ in contradiction to the assumption that $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$. We conclude that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Thus $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

Case 2.2: $f(\mathbf{x})$ is not Pareto comparable with $f(\mathbf{z})$.

Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ by Lemma 5.1.1, we have $T_{Compr}(f(\mathbf{x})) \leq T_{Compr}(f(\mathbf{y}))$. Combining with $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$, we obtain $T_{Compr}(f(\mathbf{x})) \leq T_{Compr}(f(\mathbf{z}))$, i.e., $f(\mathbf{x}) \leq_{Com} f(\mathbf{z})$.

Case 2.3: $f(\mathbf{y}) \leq_{Pareto} f(\mathbf{z})$ and $f(\mathbf{x})$ are not comparable with $f(\mathbf{y})$ with $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. The proof is similar to Case 2.1.

From Cases 1 and 2, we obtain $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

(Anti-Symmetric). Let $f(\mathbf{x}) \leq_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Compr} f(\mathbf{x})$. We must have $f(\mathbf{x}) = f(\mathbf{y})$. To obtain a contradiction, suppose that $f(\mathbf{x}) \neq f(\mathbf{y})$. Immediately we have $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$.

Case 3: $f(\mathbf{x})$ is Pareto comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$, we obtain $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. Since $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$, we obtain $f(\mathbf{y}) <_{Pareto} f(\mathbf{x})$, which contradicts the previous conclusion.

Case 4: $f(\mathbf{x})$ is not comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$, we have $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. Also, since $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$, we have $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{x}))$, contradicting the fact that $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. From Cases 3 and 4, we conclude that $f(\mathbf{x}) = f(\mathbf{y})$.

It follows that $\leq_{Compr}$ is a partial order on $f(A)$. ■

An obvious scalar equivalence of the compromise optimization problem is

$$\left\{\begin{matrix} \max\limits_{\mathbf{x}\in A} & T_{Compr}(f(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \end{matrix}\right\}.$$

5.2 An Application in Multi-objective optimization

In a Pareto maximization problem, a decision maker often selects a non-dominated point satisfying some secondary criteria such as choosing the largest summation of the objective function values. The secondary criterion here will be to select a solution equitably distributes the benefit among all objectives. Indeed, the compromise solution applied to the objective function can accomplish both the primary Pareto and secondary fairness criteria because of Lemma 5.1.2.

Consider the following Pareto maximization

$$\left\{\begin{matrix} \text{Vmax} & (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2) \\ \text{s.t.} & x_1^{\ 2} + x_2^{\ 2} \leq 1 \\ & x_1, x_2 \geq 0 \end{matrix}\right\}.$$

The Pareto frontier is shown in Figure 5.1 below.

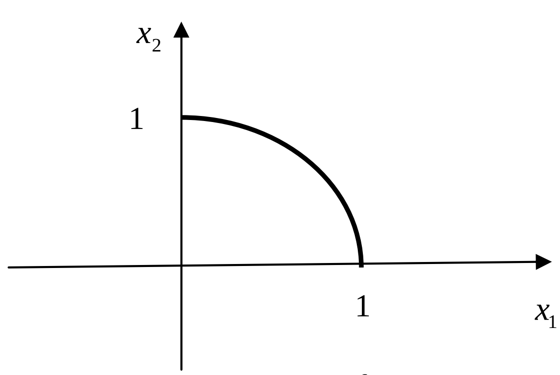

Figure 5.1 Pareto frontier.

Let $A = \{(x_1, x_2) : x_1^{\ 2} + x_2^{\ 2} \le 1, x_1, x_2 \ge 0\}$. Here, $M_1 = [\max_{\mathbf{x} \in A}(f_1(\mathbf{x}) = x_1)] = 1$,

$M_2 = [\max_{\mathbf{x} \in A}(f_2(\mathbf{x}) = x_2)] = 1$, $m_1 = [\min_{\mathbf{x} \in A}(f_1(\mathbf{x}) = x_1)] = 0$, $m_2 = [\min_{\mathbf{x} \in A}(f_2(\mathbf{x}) = x_2)] = 0$. The compromise transformation function thus becomes

$$\left\{ \begin{array}{lcc} T_{Compr}(f_1(\mathbf{x}), f_2(\mathbf{x})) & = & [(\frac{f_1(\mathbf{x}) - m_1 + 1}{M_1 - m_1 + 1}) \cdot (\frac{f_2(\mathbf{x}) - m_2 + 1}{M_2 - m_2 + 1})] \\ & = & [(\frac{x_1 + 1}{2}) \cdot (\frac{x_2 + 1}{2})] \end{array} \right\} \text{for } x \in A.$$

The compromise problem with the order $\le_{Compr}$ is thus

$$\left\{ \begin{array}{lc} \text{Compromise} & (f_1(x_1, x_2), f_2(x_1, x_2)) = (x_1, x_2) \\ \text{s.t.} & x_1^{\ 2} + x_2^{\ 2} \le 1 \\ & x_1, x_2 \ge 0 \end{array} \right\}.$$

An equivalent scalarization is as follows.

$$\left\{ \begin{array}{lc} \max & T_{Compr}(f(\mathbf{x})) = (\frac{x_1 + 1}{2}) \cdot (\frac{x_2 + 1}{2}) \\ \text{s.t.} & x_1^{\ 2} + x_2^{\ 2} \le 1 \\ & x_1, x_2 \ge 0 \end{array} \right\}.$$

The solution is $(x_1^*, x_2^*) = (0.707, 0.707)$ with objective value of 0.729.

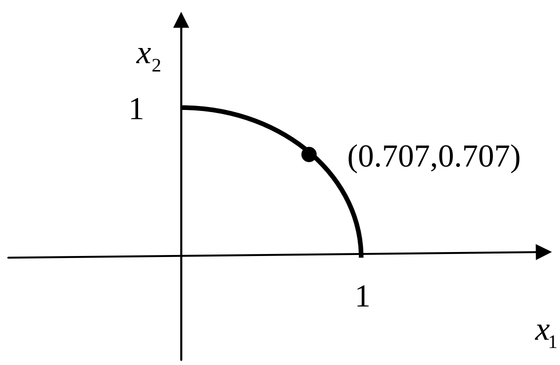

Figure 5.2 The compromise solution.

6. Conclusions and Future Work

We have unified the notion of an optimization criterion within a general axiomatic framework to include all standard optimization criteria as special cases. One requirement for an optimization criterion is the scalarizability property. Hence all optimization criteria are equivalent to solving similar scalar maximization problems, and all are equivalent in a significant sense. In particular, a scalar equivalence for GOCs has been proposed under appropriate assumptions. Examples of GOC include standard optimization criteria because they satisfy our axioms. Moreover, the group decision making of a majority voting scheme was shown not to represent an optimization criterion in our general framework. Finally, we defined a criterion based on the notion of compromising. Future work should concentrate on two areas. First, computational methods should be studied for the GSOC presented so that actual decisions can be readily made. Second, new optimization criteria should be developed to provide further models for decision making.

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